Computational Experience in Solving Equilibrium Models by a Sequence of Linear Complementarity Problems

LARS MATHIESEN

Norwegian School of Economics and Business Administration, Bergen, Norway

(Received July 1983; revised August 1984; accepted December 1984)

This paper presents a modeling format and a solution algorithm for partial and general economic equilibrium problems. It reports on computational experience from a series of small to medium sized problems taken from the literature on computation of economic equilibria. The common characteristic of these models is the presence of weak inequalities and complementary slackness, e.g., a linear technology with alternative activities or various institutional constraints on prices. The algorithm computes the equilibrium by solving a sequence of linear complementarity problems. The iterative (outer) part of this algorithm is a Newton process. For the inner part, we use Lemke's almost complementary pivoting algorithm. Theoretical results for the performance of this algorithm are at present available only for the partial equilibrium cases. Our computational experience with both types of models, however, is encouraging. The algorithm solved all nine test problems when initiated at reasonable starting points. Five of these nine problems are solved for several different starting points, indicating a large region over which the algorithm converges. Our results demonstrate that the algorithm is economical in terms of the number of pivots, function evaluations and CPU time.

WE ARE CONCERNED with solving certain types of partial and general economic equilibrium problems involving production and consumption. Because the two types of problems are mathematically similar, we find it instructive to unify their treatment within one modeling format. Our main focus, however, is on solving general (or Walrasian) equilibrium problems.

For a given problem, we formulate first order necessary conditions for each agent or sector to be in equilibrium and then observe that the resulting model is a complementarity problem, i.e.,

(CP) find $z \in R^l$ that solves $F(z) \geq 0$, $z \geq 0$ and $z^T F(z) = 0$.

It is important to distinguish between this "equilibrium" modeling approach and the "optimization" approach. The first order optimality

Subject classification: 131 economic equilibrium modeling, 622 nonlinear complementarity, 642 sequence of linear complementarity problems.

Operations Research
Vol. 33, No. 6, November–December 1985

0030-364X/85/3306-1225 $01.25

conditions of a mathematical programming model are also known to satisfy CP. Thus, if an equilibrium problem can be posed as an optimization model, it will probably be computationally more efficient to solve it by a nonlinear programming algorithm. An equilibrium model, however, may be "nonintegrable." That is, there may be no optimization model that leads to this complementarity problem. In general, lack of symmetry and/or skew-symmetry (explained below) will be the cause. Particular examples of nonintegrable equilibrium problems include those that involve: several households with distinct endowments and tastes; ad valorem taxes or tariffs; institutional constraints on prices such as minimum-wage laws; or invariant capital stock (Hansen and Koopmans 1972).

In addition to nonintegrability, we shall focus on weak inequalities and complementary slackness in these problems. The essential fact is that, for at least some of these inequalities, we do not know a priori (apart from the implications of $z^TF(z) = 0$) which will hold as strict inequalities and which will hold with equality at an equilibrium. Two classes of modeling features will result in such a model. First, inequalities are obtained if we specify the production of a commodity by alternative technical processes (typically a linear technology matrix). Second, the economic problem under consideration may involve various types of institutional constraints on prices. Some prices might be constrained by upper or lower bounds, as in fix-price models, or might be restricted by single-period budget constraints in multiperiod models.

The literature is rich in solution approaches to the computation of an economic equilibrium. The optimization approach (re-)phrases the model in terms of maximizing the sum of producers' and consumers' surplus, that is, the line integral of the inverse demand and supply functions. This idea originated with Samuelson (1947), and is widely used in economic analyses of topics such as competitive spatial equilibrium, marginal cost pricing and peak load pricing. (See e.g., Takayama and Judge 1971; Littlechild 1970; and Pressman 1970.) An optimization model can also be obtained when there are homothetic preferences. One can then postulate a single household whose utility function is representative of the economy as a whole. (See Manne 1976.) In both cases, one may compute the optimal activity levels using a nonlinear optimization algorithm, and derive market prices from the corresponding shadow prices.

A second approach, which is more desirable from an economic-theoretical point of view because it provides a constructive proof for the existence of an equilibrium, is the fixed-point method pioneered by Scarf (1973). (See also Shoven and Whalley 1972; and Shoven 1983.)

The third general approach is to employ iterative solution techniques such as tâtonnement, Gauss-Seidel, Jacobi or Newton methods. This

approach is perhaps most commonly used for solving equilibrium models phrased as a system of nonlinear (excess demand) equations.

For the model types we consider, each of these methods poses difficulties. The optimization approach (which is tailored for complementary slackness) is available only if the complementarity model is integrable. Thus, to use this approach, one must modify functions or abstain from modeling certain economic features. Nonlinear equation methods (for which integrability is of no concern) are traditionally restricted to a set of equations and not inequalities. They do not allow for bounds on individual variables, and do not easily accommodate complementary slackness conditions. The fixed-point techniques are general-purpose. They are, however, known to perform much less efficiently on point-to-set maps than on point-to-point maps. Fixed-point techniques also seem to have distinct limitations with respect to the dimensions of the problem. These considerations motivate the present work.

There are ways of dealing with each of these deficiencies. Nonintegrable models have been approximated by a sequence of integrable models and solved by standard optimization methods. (See the PIES algorithm described by Hogan 1975 and Ahn 1978. See also Carey 1977.) Pang and Chan (1982) have extended iterative methods to deal with complementary slackness. Awoniyi and Todd (1981) have accelerated fixed-point algorithms using quasi-Newton methods, and Ginsburgh and Waelbroeck (1981) and Dervis, De Melo and Robinson (1982) have developed solution approaches that combine several algorithms. These contributions also reduce, in several ways, the differences among the three approaches.

Our Sequential Linear Complementarity (SLCP) algorithm computes an equilibrium by solving a sequence of linear approximations obtained by taking first-order Taylor expansions of nonlinear terms. The resulting linear complementarity problems (LCPs) are solved by Lemke's "almost complementary pivoting method." (See Lemke 1965 or Cottle and Dantzig 1968.) Both the modeling format and this solution approach have been reported previously for partial equilibrium (PE) models (Hansen and Manne 1977, Mathiesen 1977, Eaves 1978 and Josephy 1979a, b. See also Irwin and Yang 1982 and Friesz et al. 1983 on dealing with economic spatial equilibrium problems, and Dafermos 1980 and Aashanti and Magnanti 1982 on computing an equilibrium in a traffic network.) This paper contributes to the literature by reporting on the adaptation of the CP format and SLCP algorithm to general equilibrium (GE) models, and describes a wide range of computational experiments with both types of models.

The CP format we employ treats prices and activity levels simultaneously. Consequently, the LCP matrix may be large, but typically will be sparse. Using Tomlin's (1978) implementation of Lemke's algorithm, we

exploit sparsity. Our approach contrasts with the application (to this format) of a fixed-point algorithm, which does not exploit sparsity and hence is hampered by the dimensionality of the format. (See Fisher and Gould 1974 and Todd 1980, who propose algorithms, but do not present computational results.)

The SLCP algorithm consists of two parts, an outer iterative part and an inner general-purpose matrix inversion technique (Lemke's algorithm). In the neighborhood of an equilibrium, the iterative algorithm is a Newton process. Its inner part seems to share the pathseeking capability of fixed-point techniques by means of complementary pivots. These pivots also relate the algorithm to optimization codes built around the simplex technique. (See e.g., Murtagh and Saunders 1978, 1983.) In total, the SLCP algorithm shares some features with all the above-mentioned approaches. From a computational perspective, this combination has proven to be highly effective.

Theoretical results for the convergence of this type of algorithm typically rely on the positive semi-definiteness of the gradient ∇F of the mapping F. In partial equilibrium context, ∇F typically will be positive semi-definite and convergence can be proved. We demonstrate that for the general equilibrium case, this assumption is less likely to be valid. In particular, the income effect of household demand tends to cause indefiniteness of ∇F.

In this report, we will emphasize computational experiments and attempt to demonstrate or explain why the results we found are so encouraging.

The outline of this paper is as follows. In the next section we present the modeling format and discuss some aspects of economic equilibrium modeling. Section 2 deals with the algorithm, its iterative steps, the linear complementarity problem, and some implementation issues. Thereafter, in Sections 3 and 4, we report on the numerical examples and our computational results. Section 5 concludes with some thoughts on the apparent numerical success of the SLCP algorithm.

1. ECONOMIC EQUILIBRIUM MODELING

The equilibrium problem of an economy is traditionally stated in terms of excess demand functions determined by the endowments of the economy, the preferences of its members, and its technology. To simplify the present exposition and thereby convey the essentials of the modeling format, we will restrict ourselves to an economy with competitive behavior throughout with no price distortions. Extensions to distortive ad valorem taxes, a public sector, a foreign sector with imports and exports, institutional constraints on prices, or a noncompetitive behavior, would be easy to accommodate. (See our test examples in Tables I–III; Mathiesen and Lont 1983; and Mathiesen and Steigum 1983.)

Consider an economy or a sector with production. Suppose that it has m commodities and n activities with constant returns to scale production. We base our presentation of the modeling format and the algorithm on the assumption that production is characterized by a linear technology matrix with fixed input-output coefficients. Two of the test examples have nonlinear production functions with price-responsive input coefficients. The successful application of SLCP to these two models indicates a wider applicability of SLCP than that presented below. (See Mathiesen 1985.)

For $i = 1, \cdots, m$ and $j = 1, \cdots, n$, let

$p = (p_i)$ denote the vector of prices,
$b = (b_i)$ denote the vector of endowments,
$d(p) = (d_i(p))$ denote the market demand functions, which we assume to be point-to-point and continuously differentiable,
$y = (y_j)$ denote the vector of activity levels,
$c = (c_j)$ denote the vector of unit costs of operating the activities, and finally, let
$A = (a_{ij})$ denote the technology matrix of input-output coefficients consistent with unit production, where $a_{ij} > 0$ ($a_{ij} < 0$) denotes an output (input).

Because of the generality of the theory of economic equilibrium, there are several ways to characterize an equilibrium. We shall use the following Definition (cf. Scarf, Def. 5.1.3):

A price vector p^* and a vector of activity levels y^* constitute a *competitive equilibrium* if:

No activity earns a positive profit;

$$c - A^T p^* \geq 0. \tag{1.1}$$

No commodity is in excess demand;

$$b + Ay^* - d(p^*) \geq 0. \tag{1.2}$$

No prices or activity levels are negative;

$$p^* \geq 0, \quad y^* \geq 0. \tag{1.3}$$

An activity earning a deficit is not used and an operated activity has no loss;

$$(c - A^T p^*)^T y^* = 0. \tag{1.4}$$

A commodity in excess supply has zero price, and a positive price implies market clearance;

$$p^{*T}(b + Ay^* - d(p^*)) = 0. \tag{1.5}$$

The vector c of operating costs represents factors of production that are

exogenous to the economy or sector under consideration. It typically applies in a partial equilibrium setting.

Assume that (1.1)–(1.5) describe a *general* equilibrium problem (of a closed economy). Then the cost vector $c = 0$ because all prices will be determined simultaneously and no single price will be exogeneously given. In this case, demands $d_i(p)$ for $i = 1, \cdots, m$ are functions of all prices in the economy, i.e., both product and factor prices. Furthermore, these demand functions will usually be specified in a manner consistent with individual household utility maximization, that is, $d_i(p) = \sum_h x_i^h$, where x_i^h is the hth household's utility maximizing demand of commodity i. Households' excess demands are given by $d(p) - b$. If the demands satisfy each individual household's budget and there is nonsatiation, then $p^T d(p) = p^T b$, and the demand functions $d(p)$ are homogeneous of degree 0 in all prices. We see that when $c = 0$, (1.1)–(1.5) determine only relative prices. That is, if the vector p^* represents equilibrium prices, so does λp^* for any scalar $\lambda > 0$. Hence, we are free to normalize the prices.

The *complementarity format* is as follows:

(CP) find $z \in R^l$ that solves $F(z) \geq 0$, $z \geq 0$ and $z^T F(z) = 0$.

When the mapping F of R^l into itself is an affine transformation, say $F(z) = q + Mz$, the corresponding complementarity problem is said to be linear, otherwise it is nonlinear. In the iterative process we shall solve a linear complementarity problem (LCP), denoted (q, M). The structure of the matrix M will then be of special concern, so let us partition M

$$M = \begin{bmatrix} Q & S \\ U & V \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & V \end{bmatrix} + \begin{bmatrix} 0 & S \\ U & 0 \end{bmatrix} = M_1 + M_2 \tag{2}$$

where Q and V are square matrices. If $S = -U^T$, then M_2 is called skew-symmetric. If, in addition, Q and V are symmetric (and so is M_1), then M is called bisymmetric.

The association between our equilibrium problem (1) and the complementarity format (CP) is given by

$$z = \begin{bmatrix} y \\ p \end{bmatrix} \quad \text{and} \quad F\begin{bmatrix} y \\ p \end{bmatrix} = \begin{bmatrix} c & -A^T p \\ b + Ay - d(p) \end{bmatrix}. \tag{3}$$

From (3), it is also clear that, if the market demand functions are linear in prices, i.e., $d(p) = d + Dp$, then our definition of an economic equilibrium will be a linear complementarity problem. In this case, the matrix M is bisymmetric if D is symmetric, and furthermore, M is positive semidefinite if D is negative semidefinite.

In the introduction, we distinguished between the optimization approach and the equilibrium approach. The bridge between them is the notion of a stationary point.

Let P be a convex set in R^m and $g : P \to R^m$ a map. A point $\pi^* \in P$ is defined to be a *stationary point* of the pair (g, P) if $\pi^{*T} g(\pi^*) \geq \pi^T g(\pi^*)$ for all $\pi \in P$. The following result is known:

Let $P = \{p \mid A^T p \leq c, p \geq 0\}$, i.e., a closed polyhedral, convex set and let $g(p) = d(p) - b$ be continuous. Then p^* is a stationary point of the pair $(d - b, P)$ if and only if

$$\begin{aligned} A^T p^* &\leq c, \quad p^* \geq 0, \\ Ay &\geq d(p^*) - b, \\ (c - A^T p^*)^T y = 0, &\quad p^{*T}(Ay + b - d(p^*)) = 0, \\ y &\geq 0, \end{aligned} \tag{4}$$

has a solution $y \in R^n$.

Hence p^* and y^* constitute an equilibrium (1) if and only if p^* is a stationary point of $(d - b, P)$. (Note that (4) assumes skew-symmetry.)

A stationary point corresponds to an optimization problem. In fact, if M of an LCP derived from (1) is bisymmetric, then there is an equivalent quadratic programming (QP) problem (and its dual). These QPs are:

$$\text{minimize } \{c^T y - \tfrac{1}{2} p^T D p\} \quad \text{subject to} \quad Ay + b - d - Dp \geq 0, \quad y \geq 0,$$

and

$$\text{maximize } \{p^T d + \tfrac{1}{2} p^T D p - p^T b\} \quad \text{subject to} \quad A^T p \leq c, \quad p \geq 0.$$

Awoniyi and Todd present a general framework which, among other applications, contains a "distorted stationary point problem." In our notation, this formulation is obtained by replacing the matrix A in (1.1) and (1.4) by an equally dimensioned matrix B, that is

$$c - B^T p^* \geq 0, \tag{1.1'}$$

and

$$(c - B^T p^*)^T y^* = 0. \tag{1.4'}$$

We now have a more general formulation, which includes (1.1)–(1.5) as a special case. The following two problems provide examples of nontrivial applications.

(i) Ad valorem taxes and subsidies on factors of production. Let $t_{ij} \geq 0$ be an ad valorem tax on input (or subsidy on output, $t_{ij} < 0$) of factor i into activity j. The cost coefficient b_{ij} is then defined as $b_{ij} = (1 + t_{ij}) a_{ij}$. (See test problem no. 2.)

(ii) The invariant capital stock problem. (See Hansen and Koopmans.) In this case the coefficients a_{ij} and b_{ij} are defined as $a_{ij} = -(d_{ij} - e_{ij})$ and $b_{ij} = (d_{ij} - \alpha e_{ij})$ for d_{ij} and e_{ij} nonnegative and the utility discount factor $\alpha < 1$. (See test problem no. 7.)

As evidenced by these two examples and our other test problems in Section 3, equilibrium problems will, in several instances, result in non-bisymmetric LCPs. In these cases, there are no equivalent QPs. It would be inappropriate for us to conclude that an optimization approach to the solution of these problems is not feasible, cf. the PIES algorithm and Carey's suggested parametric revisions to obtain an integrable model. Our computational experience, however, indicates that the SLCP may be a more efficient approach for several types of problems.

2. AN ITERATIVE ALGORITHM

Our SLCP algorithm computes the equilibrium prices and activity levels by solving a sequence of approximating LCPs. These linear approximations are obtained by taking first order Taylor expansions, resulting in a Newton-like iterative process. Each LCP is solved using Lemke's almost complementary pivoting algorithm.

Let ε and δ be two positive scalars of small value. The iterative algorithm to solve (1) can be summarized by the following steps:

1. Initialization:
   set $k = 0$ and stipulate z^0.
2. Iteration count:
   replace k by $k + 1$ and set $\bar{z} = z^{k-1}$.
3. Evaluate the linearization:
   $LF(z \mid \bar{z}) \equiv q^k + M^k z$.
4. Solve the LCP (q^k, M^k):
   find z and w that solve
   $w = q^k + M^k z \geq 0$, $z \geq 0$, and $w^T z = 0$.
5. Construct the iterate:
   if $z \in$ domain F and $z \in$ domain ∇F, then the iterate $z^k = z$;
   if not, construct a modified iterate (see our later discussion).
6. Test for termination:
   if $F_i(z^k) > -\delta$ and $\mid z_i^k \cdot F_i(z^k) \mid < \varepsilon$ for all i, then z^k is an approximate equilibrium. If not, return to step 2.

There are three critical points associated with the analysis of iterative processes. The first is to ascertain that the iterates $\{z^k\}$ are well defined—that is, at a given iteration, we must be able to obtain a solution to the linear approximation. However, not just any solution will do. The iterate must belong to the domains of F and ∇F. The second point concerns the convergence of the generated sequence $\{z^k\}$ and whether its limit point is a solution to our problem. Our third concern is the economy of the entire process. We now address these points, although the focus of this paper is on the third issue.

2.1. Constructing the LCP

The first order Taylor expansion of a function $f(x)$ at a point $\bar{x}$ is

$$Lf(x \mid \bar{x}) \equiv f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) = (f(\bar{x}) - \nabla f(\bar{x})\bar{x}) + \nabla f(\bar{x})x \equiv h + Hx,$$

where $H = \nabla f(\bar{x}) = (\partial f_i(\bar{x})/\partial x_j)$ is the matrix of partial derivatives of f with respect to x, and h is a vector. (In an optimization framework, F of CP would itself be a gradient mapping, say $F = \nabla\phi$, and F's gradient matrix H would be the Hessian matrix of some function $\phi(x)$.)

(1.1) is expressed in a linear form and (1.2) represents the only nonlinearities in the model. Linearizing $b + Ay - d(p) \geq 0$, we obtain

$$\begin{aligned} b + Ay - d(\bar{p}) + \nabla d(\bar{p})\bar{p} - \nabla d(\bar{p})p \\ &= b - \{d(\bar{p}) - \nabla d(\bar{p})\bar{p}\} + Ay + \{-\nabla d(\bar{p})\}p \\ &\equiv b - g + Ay + Gp, \end{aligned}$$

where g is a vector and G is the negative of the Jacobian matrix of household demand and supply.

A linear approximation of (1) can then be stated as follows:

Find activity levels y and prices p, satisfying

$$w_1 = c - A^T p \geq 0, \tag{5.1}$$

$$w_2 = (b - g) + Ay + Gp \geq 0, \tag{5.2}$$

$$y \geq 0, \quad p \geq 0, \tag{5.3}$$

$$w_1 y = 0, \tag{5.4}$$

$$w_2 p = 0. \tag{5.5}$$

(5.1)–(5.5) is the approximating LCP for a PE-model (partial equilibrium model). In a GE-case (general equilibrium case), however, (5.1)–(5.5) may not be an appropriate model. Note that when function f is homogeneous of degree zero in all arguments, then $\nabla f(x)x = 0$ for all x, implying that when $x \neq 0$, $\nabla f(x)$ is a singular matrix. In a Walrasian equilibrium model, household demands $(d(p))$ are homogeneous of degree zero in all prices. A basis matrix of the linear approximation, (5.1) and (5.2), will be singular at an equilibrium. It will be nearly singular when the process approaches equilibrium, and hence numerically unstable.

This singularity problem is, of course, nothing but the observation that in a Walrasian equilibrium, (1.1)–(1.5) determines only relative prices. We avoid this impasse by choosing a numéraire, stipulating its price, say $p_i = \bar{p}_i$ and dropping the i'th excess-supply constraint of (5.2). We thereby create an LCP of dimension $n + m - 1$. Because there are m commodities, there are m candidates for numéraire and hence m alternative LCPs to

choose among at a given iterate $\bar{z}$. Any of these can possibly be solved to obtain the next iterate. (In the sequel, it is understood that dimensions of vectors and matrices of the LCP are reduced by 1 when we discuss GE-models.) An alternative normalization that avoids the singularity is to add the constraint $ep = 1$, where $e = (1, \cdots, 1)$. This device is applied in fixed-point techniques.

In a partial equilibrium model, the "price-level" is already given by the exogeneous vector c.

2.2. Solving the LCP

In order to obtain the iterate (z^k), we apply Lemke's algorithm. This algorithm is known to terminate either with a solution to the LCP, or by finding a so-called secondary ray of almost complementary solutions. To prove that Lemke's algorithm computes a solution to a particular LCP, it therefore suffices to show that this LCP has no secondary ray.

Three of our test problems (descriptions of all test problems appear in Tables I and II, Section 3, and Table III, Section 4) have a structure such that this can be done. In a separate appendix (available upon request to the author), we have shown that LCPs of problems 2, 3 and 7 have no secondary rays. Thus Lemke's algorithm computes a solution to these LCPs. The presence of ad valorem taxes in model 2, and a discount factor in model 7, both destroying the skew-symmetry of the matrix M (see end of Section 1), do not affect this result.

Lemke has proved a general result, which applies in the following way to our situation:

> Let the matrix M be copositive plus. If Lemke's algorithm terminates in a secondary ray, then (5.1)–(5.3) has no solution. (See also Cottle and Dantzig.)

This result implies a constructive proof for the existence of a solution, that is, if there is a solution for the LCP, then Lemke's algorithm computes one. In order to prove that iterates are well-defined, however, we shall have to demonstrate that the generic LCP does indeed have a solution.

For the PIES energy model (problem 1), the Jacobian of demand is negative definite for $\bar{p}$ contained in a set of $P \leq R^6_+$. P contains most of R^6_+ and includes the equilibrium. (There are, however, price vectors $\hat{p} > 0$, such that $x^T[\nabla d(\hat{p})]x > 0$, that is, the Jacobian is not negative definite on all of R^6_+.) When $\nabla d(p)$ is negative definite, M has the copositive plus property. (See also Josephy 1979b.) Existence of a feasible solution (to (5.1)–(5.3)) follows because the function $p^TGp + p^T(b - g)$ is bounded below on the set $\pi = \{p \geq 0, A^Tp \leq c\}$. Then, according to Eaves' (1971) result, the LCP has a solution.

Thus, in four cases (examples 1–3 and 7), we obtain a solution from the linear approximation, and can construt a well-defined iterate. We observe that examples 1–3 are PE models, while the seventh is a planning type (non-Walrasian) model. In PE-models, the Jacobian matrix of household demand, $\nabla d(p)$, will typically be negative (semi-) definite—as in examples 1–3. Thus in partial equilibrium problems the matrix G will typically be positive semi-definite and hence copositive plus. Existence of feasible vectors y and p usually follows from an economically meaningful construction. If so, Lemke's algorithm will compute a solution.

Consider, then, the Walrasian model. In this case, household demands satisfy individual households' budgets and the Jacobian matrix might not be negative semi-definite. Our fifth test example illustrates this point. Here demands are given by functions

$$\begin{aligned} d_i(p) &= \textstyle\sum_h x_i^h(p) = \sum_h \dfrac{\alpha_i^h(\sum_k p_k b_k^h)}{p_i}, \quad i = 1, \cdots, 7, \\ d_i(p) &= 0, \qquad\qquad\qquad\qquad\qquad\quad i = 8, \cdots, 14, \end{aligned} \tag{6.1}$$

where $k = 11, 12, 13$; $h = 1, \cdots, 4$, and its Jacobian matrix is

$$\nabla d(\bar{p}) = \begin{bmatrix} D_1 & 0 & D_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.2}$$

$D_1 = (\partial d_i/\partial p_i)$ is a (7×7) diagonal matrix will negative entries, hence it is negative definite. $D_3 = (\partial d_i/\partial p_k)$ is a (7×3) matrix with positive entries. D_1 accounts for the (partial equilibrium) effects of the Slutsky equation of consumer demand theory. ((6.1.) is known to display gross-substitutability in this respect, and hence be represented by a negative definite Jacobian.) D_3 accounts for general equilibrium effects because income depends on endogeneous prices and is not exogeneously stipulated as in a partial equilibrium setting.

The matrix $\nabla d(\bar{p})$ is clearly not negative semidefinite and we observe that it is precisely the general equilibrium income-effects that violate the structure. This indefiniteness of the Jacobian also characterizes our other Walrasian test models 4, 6 and 9.

We are unaware of any theoretical result that can come to our rescue to prove that Lemke's algorithm will compute a solution, if one exists, for the LCP of a Walrasian model. Our empirical finding, however, is that Lemke's algorithm does indeed process the LCPs, that is, compute a solution or show that none exists. For the text examples in this report, we never experienced an LCP without a solution. Subsequent work has provided an example where an LCP of the SLCP process can have none,

one or two solutions, depending on the choice of numéraire and point of linearization (Mathiesen 1984).

Remember that, in the Walrasian case, we choose a numéraire to obtain an LCP. Hence, when $\mathrm{LCP}_i(\bar{z})$ has no solution, we can switch to numéraire j, $j \neq i$, and solve $\mathrm{LCP}_j(\bar{z})$. In the example referred to in the last paragraph, this device helped us solve the Walrasian model from any starting point on the price simplex.

The linearization implied by the first order Taylor expansion, (6.2), is

$$d(\bar{p}) + D_1 p_1 + D_3 p_3.$$

Consider the alternative linearization

$$(d(\bar{p}) + D_3 \bar{p}_3) + D_1 p_1 = 2d(\bar{p}) + D_1 p_1. \tag{7}$$

Here the Jacobian matrix is diagonal and negative semidefinite. Hence Lemke's algorithm computes a solution to the LCP, if there is one. This alternative linearization is along the lines suggested by Carey for iterative revisions of nonintegrable demand and supply functions.

The Taylor expansion provides the best approximation in terms of speed of convergence near the equilibrium. It does not necessarily do so when it is far away from the equilibrium. In that case, some other linearization might perform better. Except for a few runs that applied the alternative linearization provided by (7) to test example 5, we have not explored such possibilities. In these particular experiments, the alternative linearization required 1–2 more iterations than our standard SLCP algorithm.

2.3. The Iterative Sequence

The solution (y, p) obtained from an LCP at iteration k could be outside domain F. $p_i = 0$ for some i, $i = 1, \cdots, 7$, for example, is outside domain $d_i(\cdot)$ as defined by (6.1). Consider then the price iterate p^k, defined by

$$p^k = p^{k-1} + \gamma(p - p^{k-1}), \quad k \geq 1, \tag{8}$$

where p solves the LCP and $0 < \gamma < 1$. One could optimize the iterate with respect to the step length γ. So far we have not used this approach, but have arbitrarily used $\gamma = \frac{1}{2}$ (when z is outside domain F).

Assume that each iterate of the sequence $\{z^k\}$ is well-defined. What can we say about its convergence? Again, there are available results for the PE-models. Motivated by the PIES algorithm and energy model, Eaves (1978) has studied the application of a sequence of LCPs to this

partial equilibrium problem. In our notation, Eaves' result can be stated as follows. Let

$$F\begin{bmatrix} y \\ u \\ v \end{bmatrix} = \begin{bmatrix} c & - D^T u - E^T v \\ a + Dy & \\ b + Ey & + f(v) \end{bmatrix}. \tag{9}$$

Assume that the function f is continuously differentiable and has derivatives that are positive definite, but not necessarily symmetric. Assume further that an equilibrium (y^*, u^*, v^*) exists. If v^0 is close to v^*, then the iterates v^k generated by the iterative process converge to v^*.

The correspondence between (1) and (9) is given by

$$A = \begin{bmatrix} D \\ E \end{bmatrix}, \quad p = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad d(p) = \begin{pmatrix} 0 \\ -f(v) \end{pmatrix}.$$

Clearly, Eaves' result applies to our test problem 1 (the PIES energy model). Test examples 2 and 3 have $\nabla d(\cdot) = -\nabla f(\cdot)$ negative definite. Also, test problem 7 (the invariant capital stock problem) has a positive definite gradient.

We demonstrated above that it would be inappropriate in the Walrasian model to assume the gradient of household demand is positive semidefinite. Definiteness seems to be too strong an assumption to make when income distribution is exogenous. Our computational results, which indicate global convergence for the Walrasian models numbers 4 and 5, seem to necessitate a different set of assumptions.

Pang and Chan have developed a convergence theory for various linear iterative algorithms to solve nonlinear complementarity problems. They consider four approaches to establish the desired convergence. Some of these results are applicable to the PE models, but none seems immediately relevant to the Walrasian problem. Pang and Chan discard the symmetry approach because the gradient matrix $\nabla F(\bar{z})$ usually is not symmetric in equilibrium models. We also observe that $\nabla F(\bar{z})$ is not positive definite as required by the norm-contraction method, nor is it an H-matrix with positive diagonals as required by the vector-contraction method, and finally, $F(z)$ is concave and not convex (as required by the monotone approach).

Assume that the process converges and terminates at iteration K. By construction, for example by solving the LCP at iteration K, $z^K \geq 0$. Next, by the termination tests, $F_i(z^K) > -\delta$ and $z_i F_i(z^K) < \varepsilon$ for all i. Hence the conditions $F(z) \geq 0$ and $z^T F(z) = 0$ are approximately satisfied, and z^K is an approximate equilibrium.

2.4. Computational Effort

The cost of the iterative algorithm is the main focus of this paper and will be reported in the next sections. In this section, we make several brief comments.

In the final iterations (near the equilibrium), the set of basic variables typically remains the same. Thus, in iteration k, we invert the matrix (B) corresponding to the same basic variables as in the previous iteration. The computational work involved in this inversion is, because of factorization of the basis, less than was necessary to execute Lemke's algorithm. If $B^{-1}q^k$ has negative components, showing that the chosen basis is not feasible, we apply Lemke's method to the transformed problem $(B^{-1}q^k, B^{-1}M^k)$. That is, we restart Lemke's method at this solution instead of at the origin. The rationale is that the solution of the preceding LCP is close to the solution of the present. Our basis for doing so is the result.

> If M is positive semidefinite, so is every principal transform of M. (See Cottle and Dantzig.)

The operations involved in inverting B amount to a sequence of principal transforms of M. As noted above, M might not be positive semidefinite. In such cases, we can only hope that the desired property carries over. That is, if Lemke's method computes a solution to (q, M), then it also computes a solution to $(B^{-1}q, B^{-1}M)$. This result turned out to hold for every run performed for this report.

Our program uses Tomlin's (1978) program LCPL. (See also Tomlin 1976a, b.) His code is based on sparse matrix factorization and updating techniques. It allows the user to specify a partial basis which the program completes and uses as the initial basic solution. We have stripped off all input and output routines and written our own interface, which essentially avoids writing (and reading) each LCP in MPS-format onto some intermediate file.

3. NUMERICAL TEST PROBLEMS

Eight of our test problems are taken from the existing literature on the computation of equilibria. The ninth problem is an application to the Norwegian economy. These problems include 3 partial and 6 general equilibrium models of which 4 are of Walrasian type. They range from small (13–14 variables) to medium-sized (160–170 variables). The common feature of these models is their inequalities, which result mainly from alternative activities (8 out of 9 problems), but also from institutional constraints on prices (2 problems).

Tables I and II present some statistics on the test problems and the resulting LCP-formulations. A more detailed description of the problems

TABLE I

NUMERICAL TEST PROBLEMS

Problem No.	Name and Reference	Dimensions[a] (n, m, h)	Demand System	Technology	Special Features
1	PIES, Hogan (1975)	(38, 6, ·)	$q_i^0 \Pi_j (p_j/p_j^0)^{e_{ij}}$	Leontief	
2	Generalized transportation, MacKinnon (1976)	(3 × 5, 3 + 5 + 1, ·)	$a_i + b_i p_i^{e_i}$	Leontief	Variable transportation costs. Ad valorem taxes
3	Generalized von Thünen, MacKinnon (1976)	(4 × 15, 4 + 15 + 1, ·)	$a_i Y/p_i$	Cobb-Douglas	
4	General equilibrium, Scarf and Hansen (1973) (pp. 109–113)	(8, 6, 5)	Derived from CES utility functions	Leontief	Relative prices only
5	General equilibrium, Scarf and Hansen (1973) (pp. 113–119)	(26, 14, 4)	Derived from Cobb-Douglas utility functions	Leontief	Relative prices only
6	2RT, Manne and Preckel (1982)	(4T, 7T + 2, 2)	Derived from discounted intertemporal logarithmic utility functions	Cobb-Douglas	Balance of payments constraints. Relative prices only
7	Invariant capital stock, Hansen and Koopmans (1972)	(10, 4, ·)	Gradient of a utility function	Leontief	
8	Institutional price constraints, (10, 8, ·) Mathiesen and Hansen (1980)	(10, 8, ·)	Linear in all prices	Leontief	Ad valorem taxes, public budget, bounds on prices, transfer to unemployed
9	NORGE, Rutherford (1982)	(102, 71, 1)	Derived from Cobb-Douglas utility function	Leontief with sector specific capital	Ad valorem taxes, public sector transfers to households. Relative prices only

[a] (n, m, h) refer to the number of activities, prices (= commodities) and households respectively. The dots for several models indicate that a system of demand equations is specified directly.

TABLE II

STRUCTURE AND SIZE OF LCP(k)

Problem No.	Structure $(q, M)^a$	Size		Density %		
		Submatrices	M	A	D	M
1	$\begin{pmatrix} c \\ b - \hat{d} \end{pmatrix}, \begin{pmatrix} 0 & -A' \\ A & \hat{G} \end{pmatrix}$	A is 24×20 G is 24×24 $(6 \times 6)^b$	44	14	3	8
2	$\begin{pmatrix} c \\ b - \hat{d} \end{pmatrix}, \begin{pmatrix} 0 & -B' \\ A & \hat{G} \end{pmatrix}$	A, B are 9×15 G is 9×9	24	33	11	17
3	$\begin{pmatrix} c \\ b - \hat{d} \end{pmatrix}, \begin{pmatrix} 0 & -\hat{A}' \\ \hat{A} & \hat{G} \end{pmatrix}$	A is 20×60 G is 20×20 (5×5)	80	15	6	6
4	$\begin{pmatrix} a \\ b - \hat{d} \end{pmatrix}, \begin{pmatrix} 0 & -A' \\ A & \hat{G} \end{pmatrix}$	A is 5×8 G is 5×5	13	73	80	46
5	$\begin{pmatrix} a \\ b - \hat{d} \end{pmatrix}, \begin{pmatrix} 0 & -A' \\ A & \hat{G} \end{pmatrix}$	A is 13×26 G is 13×13	39	44	17	23
6	$\begin{pmatrix} a \\ b - \hat{d} \end{pmatrix}, \begin{pmatrix} 0 & -\hat{A}' \\ \hat{A}' & \hat{G} \end{pmatrix}$	A is $(7T + 1) \times 4T$ G is $(7T + 1) \times (7T + 1)$	$11T + 1$	$T = 10$: 6	10	11
7	$\begin{pmatrix} \hat{v} \\ b \end{pmatrix}, \begin{pmatrix} \hat{V} & -B' \\ A & 0 \end{pmatrix}$	A, B are 4×10 V is 10×10 (6×6)	14	100	36	59
8	$\begin{pmatrix} 0 \\ b - \hat{d} \end{pmatrix}, \begin{pmatrix} 0 & -B' \\ \hat{A} & \hat{G} \end{pmatrix}$	A, B are 9×10 G is 9×9	19	42	31	32
9	$\begin{pmatrix} a \\ b - \hat{d} \end{pmatrix}, \begin{pmatrix} 0 & -B' \\ A & \hat{G} \end{pmatrix}$	A, B are 71×102 G is 71×71	173			

[a] The circumflex over an array indicates that some or all of its coefficients are reestimated in each iteration. Only problems 2 and 7 have symmetric Jacobian matrices G and since $B \neq A$, none of the 9 models have a bisymmetric M. In all models $-\nabla d(\bar{p})$ is included in G. Also, for problems 3 and 6, the Jacobian of input to production is included.

[b] Demand for several commodities is constant, hence non-zero coefficients constitute only a 6×6 submatrix.

and specific features of their implementations is given in an Appendix which is available upon request from the author. Except for one model, the demand functions are nonlinear. Two models have nonlinear (Cobb-Douglas) production functions. Their input-coefficients and the demand functions in model 5 (derived from CES utility functions) are differentiated numerically. In the other cases, we employ analytic gradients. Observe that none of the LCPs are bisymmetric. Hence these problems cannot be formulated as optimization models.

Because of the homogeneity of the demand functions in models 4–6 and 9, we have stipulated a numéraire. This operation converts a column of the matrix to a fixed vector, and eliminates one constraint from the original system.

4. COMPUTATIONAL EXPERIENCE

In this Section we report on our computational experience in solving 9 test problems, and compare it with that of other researchers.

All 9 problems were solved successfully. When initiated at a reasonable starting point, the process typically converged in 4 to 6 iterations. (See Table III.) The number of function and gradient evaluations and pivots are supplementary measures of computational effort. Problem 5, for example, has 7 demand functions with 7 diagonal and 21 nondiagonal elements of the corresponding gradient which must be evaluated in each iteration. The generalized von Thünen problem (problem 3) has 5 demand and supply functions which required 5 functions and 5 diagonal gradient elements to be evaluated. In addition, there are 2 price-dependent input coefficients per activity. This problem structure implied 120 evaluations to compute the production input coefficients and (a maximum) of 5×120 evaluations to compute the coefficients of the gradient matrix.

The CPU time refers to computations on a DEC-2060 (TOPS 20 version 5) with pre-COMPILED (FORTRAN version 6) and LOADed, but not OPTIMIZEd programs using the RUN command. Input resided on files, and there was a fair amount of run statistics written to files during execution.

Problems 1 and 8 were solved (partly manually) before the iterative process was programmed. The iterates were observed to converge, and the processes were terminated after 3 iterations.

Problems 2 and 3 were constructed and solved by MacKinnon (1976) using a fixed-point method on an IBM 370/91 in 0.25 and 2.3 seconds CPU time, respectively. Our solutions are accurate to the same number of digits that he reports, and the CPU times indicate computational effort of the same order of magnitude. Note that the two problems are decomposable, allowing MacKinnon to search for an equilibrium in 4

TABLE III

RUN STATISTICS

Problem No.	Initial Price Vector	No. of Iterations	No. of Function Evaluations per Iteration	No. of Pivots in Each Iteration	Accuracy ε	CPU seconds DEC 2060
1	6 demand prices (16, 16, 12, 12, 12, 12) Hogan (1975)	3[c]	6 + 18	—	10^{-3}	—
2	9 demand and supply prices (1, 1, ···, 1)	4	9 + 9	19, 15 + 2, 16, 16	$5 \cdot 10^{-6}$	1.0
3	5 demand and supply prices (1, 1, 1, 1, 1)	6	(5 + 5) + 120(1 + 5)[a]	72, 17 + 17, 27 + 18, 34 34 + 3, 36	$5 \cdot 10^{-6}$	4.6
4	6 prices (1, 1, ···, 1)	4	6 + (6 × 6)	12, 12, 12, 9	$5 \cdot 10^{-5}$	1.7
5	14 prices (1, 1, ···, 1)	4	7 + 28	43, 25 × 3	$5 \cdot 10^{-5}$	2.3
6	See Manne and Preckel (1982)					
	$T = 5$	5	—[b]	113, 57, 56 × 3		9
	$T = 10$	5	—	213, 108, 107 × 3	$5 \cdot 10^{-5}$	39
	$T = 15$	6	—	378, 158, 157 × 4		111
7	10 activity levels (0.05, 0.05, ···, 0.05)	4	6 × 4	8 + 8 × 3	$5 \cdot 10^{-6}$	0.9
8	9 prices and subsidies (1, 1, 1, 2, 1, 2, 1, 0, 0)	3[c]	—	—	10^{-3}	—
9	173 prices and rents (1, 1, ···, 1)	—	—	—	—	90

[a] Evaluation of the Jacobian of production typically required 20% of this number.

[b] The evaluations and numerical differentiations are described by a large FORTRAN subroutine, which makes the statistic uninteresting in this case.

[c] For these models a manually guided iterative process was run.

supply-prices (problem 2) and 5 product and factor prices (problem 3), compared to our 24 and 80 dimensional LCPs. This decomposition certainly accounts for his favorable results. The definitive advantage of MacKinnon's approach is the proof for global convergence. We have been able to prove only local convergence, although global convergence is indeed suggested by our computational experiments.

Rowse (1981, 1982) reports on efforts to solve these two problems by an optimization approach through the nonlinear programming code MINOS. (See Murtagh and Saunders 1978.) While his CPU times on both problems include overhead from using a standard package, they indicate that this optimization approach is inefficient compared to MacKinnon's and our results.

Scarf (1973) presents various implementations of the fixed point method for solving Walrasian equilibrium problems such as problems 4 and 5. The first implementation searches on a simplex of the full set of prices. Even with newer and more efficient codes with a restart and grid refinement procedure, this approach is inferior to ours for such "well-behaved" problems. See Scarf (1981).

Several researchers have observed that fixed-point techniques, even with grid refinement, converge too slowly when applied to point-to-set maps. Improvements are obtained by switching to a Newton-type iterative process when the set of active constraints (or basic variables) remains constant from one iteration to the next. Awoniyi and Todd applied such an accelerated fixed-point algorithm to problems 4 and 5. They reduced the CPU time for problem 5 from 14 to 5–8.5 seconds on an IBM 370/165. These execution times and their other run statistics indicate that the SLCP algorithm is more efficient, perhaps by as much as a factor of 3.

Scarf's second implementation decomposes the problem (Scarf, ch. 5.5). The search on the simplex is now restricted to only prices of those commodities that are demanded by the households. For problem 5, this amounts to 7 out of 14 prices. The tactic almost halved the CPU time. Other researchers have appealed to a "nonsubstitution" theorem. At the cost of additional side calculations, they have been able to search on a simplex of only resource prices. This approach makes rather detailed analyses feasible. (See Shoven.) The CPU time, however, may still be high, as experienced by Sierra (1982) in an analysis of the Mexican economy. Using Merrill's (1972) code, his fixed-point approach required 15 minutes on a DEC 2060 computer while a Newton process, applied to a specified set of equations, did the job in 2 minutes.

An equilibrium price vector must be nonnegative and also satisfy the nonprofitability constraints $A^T p \leq 0$. This condition is the basis for Scarf's third implementation (Scarf, ch. 6.4). Scarf obtains equilibrium prices and activity levels by solving a pure trade model with as

many commodities as there are extreme points $\pi^1, \cdots, \pi^N$ of $\{A^T\pi \leq 0, \pi \geq 0\}$ intersected with the unit simplex.

Looking now in other quarters of the literature we find only Manne, Chao and Wilson (1980). They report that their MCW-algorithm (a sequence of linear programming problems), used 22.7 seconds CPU time on an IBM 370/168 computer to solve problem 5.

Manne and Preckel (1982) constructed the 2RT model, a two-region multiperiod neoclassical Walrasian equilibrium model with nonlinear production. They report using 10, 15 and 32 minutes for solving, respectively, 9, 11, 13 time period models with Merrill's fixed-point code on a DEC 2060 computer. Our results are obtained on the same computer and indicate an order of magnitude saving for 10 period models (and possibly two orders of magnitude for 15 time period models). Note that they solved the problem in reduced form with the fixed point code searching on a 3T dimensional price simplex. Our system uses a structural form of the model and involves on the order of 11T prices and activities.

Preckel (1983) has compared 4 algorithms applied to the 2RT model: a fixed-point algorithm (Broadie's (1983) OCTASOLV), a quasi-Newton process, the SLCP algorithm and the MCW-algorithm. For $T = 3$ time periods, the first three algorithms solved their respective models in about the same CPU time and to approximately the same accuracy. Based on his reported CPU times for $T = 3$, 6 and 9 time periods, we have fitted the function

$$\text{CPU-seconds} = aT^b,$$

where a and b are parameters. The results are

Fixed-point:	$a = 0.07$,	$b = 3.8$.
Quasi-Newton:	$a = 0.28$,	$b = 2.6$.
LCP:	$a = 0.05$,	$b = 1.9$.

Three observations are, of course, too few to draw firm conclusions, but these results are suggestive of the relative merit of the three approaches applied to this particular type of equilibrium problem. (The MCW-algorithm both failed to obtain the same accuracy as the other codes for $T = 3$ and used considerably longer CPU time.)

Dantzig and Manne (1974) solved the invariant capital stock problem by approximating the utility function by a piecewise linear function and formulating a linear complementarity problem directly. They based the grid points for this approximation on the solution reported by Hansen and Koopmans. The LCP had 32 columns and rows and was solved once in 2.5–3 seconds on an IBM 360/67 computer. It is safe to conclude that iterative linearization (yielding 14 dimensional LCPs) is more efficient and accurate than a once-and-for-all piecewise linear approximation. Josephy (1979a) has solved this model by an algorithm similar to ours.

For our 9th test problem, the NORGE model, there are no comparable results available from other algorithms. Because of several types of specific features, the solution procedure requires a considerable amount of computation. Nonetheless, the execution times are only 90 seconds from a cold start (all variables at their base year levels) and 15 seconds from a warm start (the solution of a similar scenario).

In some respect NORGE is larger than Sierra's Mexico model. Sierra's experience of 2 minutes (on the same computer) with a Newton-process and from a warm start is therefore puzzling. There is, however, a basic difference between the two systems of equations to which the algorithms are applied. Sierra's approach applies to a subset of the commodity constraints, (1.2), and makes several side calculations. The CP format encompasses both commodity and nonprofitability constraints. It is a larger system, but it also addresses all relevant information simultaneously. The SLCP iterates might therefore provide more accurate approximations to the entire model, and this property might explain why it converges faster.

The computing times for these algorithms are sensitive to the arbitrary choice of starting point. On problem 2, price vectors $(c, c, \cdots, c)$ with c ranging from 0.01 to 90 were used. The number of iterations ranged between 4 and 10, and increased with extremely low or high values of c. The CPU time ranged between 1 and 1.8 second, indicating 0.14 second for each of the last iterations. Similar results are obtained with problem 3.

General equilibrium problems 4 and 5 are solved for widely differing starting points. On problem 4, initial prices $p_j^0 = 0.95$, $p_i^0 = 0.01$, for all i, $i \neq j$, and for $j = 1, 2, \cdots, 6$ were used. For $j = 2, \cdots, 5$ the processes terminated after 5 or 6 iterations, and for $j = 1$ and 6, they took 10 iterations. Each additional iteration required approximately 0.2 second. These initial prices are close to the vertices of the price simplex and are thus similar to the vertex-start used by Scarf in the early versions of fixed-point algorithms. Our experience with model 5 is analogous.

Later experiments with small-scale Walrasian equilibrium problems such as Scarf's (1960) unstable exchange equilibrium, Kehoe's (1980) multiple equilibrium model and others, also indicate convergence over large regions. (See Mathiesen and Rutherford 1983.) Scarf's exchange model, however, provides a counterexample to conjectured global convergence. That is, there are starting points $\bar{p}$ for which no $\mathrm{LCP}_i(\bar{p})$ for $i = 1, \cdots, m$ has a solution. Hence the SLCP algorithm fails to compute the next iterate. (See Mathiesen 1985.)

5. CONCLUDING REMARKS

We have presented the complementarity problem as a modeling format for partial and general equilibrium problems. This format includes as

special cases the computable general equilibrium (CGE) problem (a formulation based on neoclassical utility and production functions), the activity analysis general equilibrium (AGE) of Ginsburgh and Waelbroeck, and mixed models with an activity analysis specification of a particular sector embedded within a neoclassical formulation of the remaining economy. The format also allows for additional "system constraints" such as single-period balance of payment constraints (the 2RT-model) or minimum wages and interest rates (Mathiesen and Steigum).

The SLCP algorithm solves these models as a sequence of linear complementarity problems. We have reported on our computational experience in applying this algorithm to 9 different equilibrium models whose common characteristics were nonintegrability and inequalities. Inequalities result mainly from alternative production activities (8 of 9 models), but also from institutional constraints on prices (2 models). All models involve nonlinear demand or production functions.

The algorithm has successfully computed the equilibrium for all 9 problems. Our computational experience comprises several runs with each of these models. By any measure, the computational effort is low, compared to other algorithms applied to the same problems. Furthermore, the CPU time increases only modestly when initiating the process at starting points far removed from the equilibrium. The accuracy of an approximate equilibrium is easily improved, since the SLCP algorithm is equivalent in the tail to a Newton process.

Based on reasonable assumptions regarding the structure of a partial equilibrium model, local convergence of the SLCP algorithm can be proved. For the general equilibrium model, the traditional assumption of a negative (semi)-definite Jacobian matrix is questionable. We have not been able, however, to present alternative theoretical results that support our computational experience with these models.

Scarf (1973) discusses several intuitive approaches to the computation of equilibria and point out their pitfalls. Scarf's fixed-point code and its descendents are guaranteed to compute an (approximate) equilibrium if one exists. More recent developments such as acceleration schemes (Awoniyi and Todd), and decomposition (Shoven, and Sierra), increase the efficiency of the fixed-point codes. In view of our computational results, however, it seems that these codes still require too many iterations and hence too many function evaluations and pivots. The small steps that underlie the proof also inhibit efficiency. Needless to say, our algorithm requires more structure, namely that functions are continuously differentiable. This feature, however, is usually found in the empirical equilibrium models to which fixed-point codes are applied.

Some researchers have suggested that the complementarity format involves too large a model. Scarf (ch. 6.4) showed that the explicit recognition of the dual cone of the production set significantly improved

the efficiency of a fixed point technique. Our computational experience indicates a similar conclusion. The advantage, in terms of an improved iterate, from simultaneously solving the entire set of (linearized) equilibrium conditions seems to outweigh the added computational burden. Our results on the "NORGE" and the "2RT" models, compared to Sierra's and Preckel's work, suggest that further research into this and similar algorithms will prove worthwhile.

We know that the SLCP algorithm does not converge globally for an arbitrary Walrasian model when initiated at any starting point; this convergence property is reserved for fixed-point algorithms. In view of the indications of global convergence on problems 4 and 5, however, theoretical results on the performance of the algorithm would be of interest.

ACKNOWLEDGMENTS

This paper is based on work done while the author was visiting scholar in the Department of Operations Research at Stanford University during the academic year 1981/82. The author would like to thank members of the Department, in particular Professor Alan S. Manne, for their hospitality. Professors R. B. Cottle, C. B. Eaves and A. S. Manne gave valuable suggestions during the work and to a previous version of this paper. Helpful comments from my colleague K. P. Hagen, the editor, and an anonymous referee are also acknowledged. As usual the author is responsible for remaining errors.

The computational experiments on problem 1, Table III, were done by Mrs. Anna Mette Fuglseth.

REFERENCES

AASHANTI, H. Z., AND T. L. MAGNANTI. 1982. A Linearization and Decomposition Algorithm for Computing Urban Traffic Equilibria. Proceedings of the IEEE Large Scale Systems Symposium (October), pp. 8–19.

AHN, B. H. 1978. Computation of Market Equilibria for Policy Analysis: The Project Independence Evaluation System Approach. Ph.D. Dissertation, Stanford University.

AWONIYI, S. A., AND M. J. TODD. 1981. An Efficient Simplicial Algorithm for Computing a Zero of a Convex Union of Smooth Functions. Presented at the NBER Conference on Applied General Equilibrium Modeling, August 24–28.

BROADIE, M.N. 1983. OCTASOLV User's Guide. Tech. Rep. Dept. of Operations Research, Stanford University.

CAREY, M. 1977. Integrability and Mathematical Programming Models: A Survey and a Parametric Approach. *Econometrica* **45,** 1957–1976.

COTTLE, R. W., AND G. B. DANTZIG. 1968. Complementary Pivot Theory of Mathematical Programming. *Linear Algebra Its Appl.* **1,** 103–125.

DAFERMOS, S. 1980. Traffic Equilibrium and Variational Inequalities. *Trans. Sci.* **14,** 42–54.

DANTZIG, G. B., AND A. S. MANNE. 1974. A Complementarity Algorithm for an Optimal Capital Path with Invariant Proportions. *J. Econ. Theory* **9,** 312–323.

DERVIS, K., J. DE MELO AND S. ROBINSON. 1982. *General Equilibrium Models for Development Policy* (A World Bank Research Publication). Cambridge University Press, New York.

EAVES, B. C. 1971. The Linear Complementarity Problem. *Mgmt. Sci.* **17,** 612–634.

EAVES, B. C. 1978. A Locally Quadratically Convergent Algorithm for Computing Stationary Points. Tech. Rep. SOL 78-13, Systems Optimization Laboratory, Dept. of Operations Research, Stanford University.

FISHER, M. L., AND F. J. GOULD. 1974. A Simplicial Algorithm for the Nonlinear Complementarity Problem. *Math. Program.* **6,** 281–300.

FRIESZ, T. L., R. L. TOBIN, T. E. SMITH AND P. T. HARKER. 1983. A Nonlinear Complementarity Formulation and Solution Procedure for the General Derived Demand Network Equilibrium Problem. *J. Region. Sci.* **23,** 337–359.

GINSBURGH, V., AND J. WAELBROECK. 1981. *Activity Analysis and General Equilibrium Modelling.* North-Holland, Amsterdam.

HANSEN, T., AND T. C. KOOPMANS. 1972. On the Definition and Computation of a Capital Stock Invariant under Optimization. *J. Econ. Theory* **5,** 487–523.

HANSEN, T., AND A. S. MANNE. 1977. Equilibrium and Linear Complementarity: An Economy with Institutional Constraints on prices. In *Equilibrium and Disequilibrium in Economic Theory,* G. Schwödiauer, (ed.). D. Reidel Publ. Co., Hingham, Mass.

HOGAN, W. W. 1975. Energy Policy Models for Project Independence. *Comput. Opns. Res.* **2,** 251–271.

IRWIN, C. L., AND C. W. YANG. 1982. Iteration and Sensitivity for a Spatial Equilibrium Problem with Linear Supply and Demand Functions. *Opns. Res.* **30,** 319–335.

JOSEPHY, N. H. 1979a. Newton's Method for Generalized Equations. MRC Tech. Rep. 1965, University of Wisconsin.

JOSEPHY, N. H. 1979b. A Newton Method for the PIES Energy Model. MRC Tech. Rep. 1971, University of Wisconsin.

KEHOE, T. J. 1980. An Example of Non-uniqueness of Equilibria. MIT, Cambridge, Mass.

LEMKE, L. E. 1965. Bimatrix Equilibrium Points and Mathematical Programming. *Mgmt. Sci.* **11,** 681–689.

LITTLECHILD, S. C. 1970. Marginal Cost Pricing with Joint Costs. *Econ. J.* **80,** 323–325.

MACKINNON, J. 1976. A Technique for the Solution of Spatial Equilibrium Models. *J. Region. Sci.* **16,** 293–307.

MANNE, A. S. 1976. ETA: A Model for Energy Technology Assessment. *Bell J. Econ. Mgmt. Sci.* **7,** 379–406.

MANNE, A. S., AND P. V. PRECKEL. 1982. North-South Trade, Capital Flows and Economic Growth: An Almost Neoclassical Model. Department of Operations Research, Stanford University.

MANNE, A. S., H. CHAO AND R. WILSON. 1980. Computation of Competitive Equilibria by a Sequence of Linear Programs. *Econometrica* **48,** 1595–1615.

MATHIESEN, L. 1977. Efficiency Pricing in a Linear Programming Model: A Case with Constraints on Dual Variables. (Tech. Rep. SOL 74-18, Stanford University, 1974). Revised in *Scand. J. Econ.* **4,** 468–477.

MATHIESEN, L. 1984. The SLCP Algorithm Applied to a Walrasian Equilibrium Model: A Worked Example. Mimeo, Norwegian School of Economics and Business Administration (May).

MATHIESEN, L. 1985. Computation of Economic Equilibria by a Sequence of Linear Complementarity Problems. Mathematical Programming Study. Forthcoming.

MATHIESEN, L., AND T. HANSEN. 1980. An Equilibrium Model for an Open Economy with Institutional constraints on Factor Prices. In *Numerical Solution of Highly Nonlinear Problems*, W. Forster (ed.). North Holland, Amsterdam.

MATHIESEN, L., AND A. LONT. 1983. Modeling Market Equilibria: An Application to the World Steel Market. Working Paper MU04, Center for Applied Research, Norwegian School of Economics and Business Administration.

MATHIESEN, L., AND T. RUTHERFORD. 1983. Testing the Robustness of an Iterative LCP Algorithm for Solving Walrasian Equilibrium Models. Discussion Paper 0883, Norwegian School of Economics and Business Administration.

MATHIESEN, L., AND E. STEIGUM. 1983. Computation of Unemployment Equilibria in a Two-country Multi-period Model with Neutral Money. Discussion Paper 1083, Norwegian School of Economics and Business Administration.

MERRILL, O. H. 1972. Applications and Extensions of an Algorithm that Computes Fixed Points of Certain Upper Semi-continuous Point to Set Mappings. Ph.D. dissertation, University of Michigan.

MURTAGH, B. A., AND M. A. SAUNDERS. 1978. Large Scale Linearly Constrained Optimization. *Math. Program.* **14,** 41–72.

MURTAGH, B. A., AND M. A. SAUNDERS. 1983. MINOS 5.0 USER'S GUIDE. Technical Report SOL 83-20, Department of Operations Research, Stanford University.

PANG, J. S., AND D. CHAN. 1982. Iterative Methods for Variational and Complementary Problems. *Math. Program.* **24,** 284–313.

PRECKEL, P. V. 1983. Intertemporal Equilibrium Models: Developments and Results. Ph.D. thesis, Department of Operations Research, Stanford University.

PRESSMAN, I. 1970. A Mathematical Formulation of the Peak-load Problem. *Bell J. Econ. Mgmt. Sci.* **1,** 304–326.

ROWSE, J. 1981. Solving the Generalized Transportation Problem. *Region. Sci. Urban Econ.* **11,** 57–68.

ROWSE, J. 1982. On the Solution of Spatial Equilibrium Models. *Decision Sci.* **13,** 619–637.

RUTHERFORD, T. 1982. Computing General Equilibria in a Complementarity Format. NORGE: A General Equilibrium Model of the Norwegian Economy. Engineer degree thesis, Department of Operations Research, Stanford University.

SAMUELSON, P. A. 1947. *Foundations of Economic Analysis.* Harvard University Press, Cambridge, Mass.

SCARF, H. E. 1960. Some Examples of Global Instability of the Competitive Equilibrium. *Int. Econ. Rev.* **1,** 157–171.

SCARF, H. E. 1981. The Computation of Equilibrium Prices. In *Handbook of Mathematical Economics,* K. Arrow and M.D. Intriligator (eds.). North Holland, Amsterdam.

SCARF, H. E., with the collaboration of T. HANSEN. 1973. *Computation of Economic Equilibria.* Yale University Press, New Haven, Conn. 1983.

SHOVEN, J. B. 1983. The Application of Fixed Point-methods to Economics. In *Homotopy Methods and Global Convergence* (NATO Conference Series II: Systems Science), B. C. Eaves, F. J. Gould, H. O. Peitgen and M. J. Todd (eds.). Plenum Press, New York.

SHOVEN, J. B., AND J. WHALLEY. 1972. A General Equilibrium Calculation of the Effects of Differential Taxation of Income from Capital in the U.S. *J. Public Econ.* **1,** 281–321.

SIERRA, H. F. 1982. Price Rigidities in a General Equilibrium Model for the Mexican Economy. Ph.D. thesis, Department of Operations Research, Stanford University.

TAKAYAMA, T., AND G. G. JUDGE. 1971. Spatial and Temporal Price and Allocation Models. North-Holland, Amsterdam.

TODD, M. J. 1980. A Quadratically-Convergent Fixed-point Algorithm for Economic Equilibria and Linearly Constrained Optimization. *Math. Program.* **18,** 111–126.

TOMLIN, J. A. 1976a. USER'S GUIDE FOR LCPL. A program for Solving Linear Complementarity Problems by Lemke's Method. Tech. Rep. SOL 76–16, Systems Optimization Laboratory, Stanford University.

TOMLIN, J. A. 1976b. PROGRAMMER'S GUIDE TO LCPL. A Program for Solving Linear Complementarity Problems by Lemke's Method. Tech. Rep. SOL 76-25, Systems Optimization Laboratory, Stanford University.

TOMLIN, J. A. 1978. Robust Implementation of Lemke's Method for the Linear Complementarity Problem. *Math. Program. Study* **7,** 55–60.