Conjugate Duality and CES Technology/Preferences

Thomas F. Rutherford

Department of Agricultural and Applied Economics
University of Wisconsin, Madison

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The Ordinary Demand Function

The problem of preference maximization can be written as

$$\max u(\mathbf{x})$$

such that

$$\mathbf{px} \leq m$$

$$x \geq 0$$

The **demand function** relates commodities prices **p** and m to the demanded bundle. We denote this as $\mathbf{x}(\mathbf{p}, m)$.
When preferences are *strictly convex*, there is a *unique* bundle which maximizes utility.

Indirect Utility

Provided that preferences are well behaved, the consumer's value-function can be written as:

$$v(\mathbf{p}, m) = \max u(\mathbf{x})$$

such that

$$\mathbf{px} = m$$

We refer to $v()$ as the *indirect utility function*.

Properties of Indirect Utility

1. $v(\mathbf{p}, m)$ is *nonincreasing* in $\mathbf{p}$ and *nondecreasing* in $\mathbf{m}$. In other words, "increasing prices cannot be good" while "increasing income cannot be bad".
2. $v(\mathbf{p}, m)$ is *homogeneous of degree 0* in (p, m). Proportional scaling prices and income does not change demand. This is a model which depends on relative rather than absolute price level.
3. Provided that the underlying preferences are convex, the indirect utility function $v(\mathbf{p}, m)$ is quasiconvex in $\mathbf{p}$.
4. $v(\mathbf{p}, m)$ is *continuous* for all nonzero prices and income.

The Expenditure Function

Provided that welfare is monotonically increasing in income, we can invert the *indirect utility* to obtain the *expenditure function*:

$$e(\mathbf{p}, u) = \min \mathbf{px}$$

such that

$$u(\mathbf{x}) \geq \bar{u}$$

The expenditure function relates the minimum cost of achieving a fixed level of utility ($\bar{u}$).

Properties of the Expenditure Function

1. $e(\mathbf{p}, u)$ is *nondecreasing* in $\mathbf{p}$. Increasing prices cannot reduce cost.
2. $e(\mathbf{p}, u)$ is *homogeneous of degree 1* in $\mathbf{p}$.
3. $e(\mathbf{p}, u)$ is *concave* in $\mathbf{p}$.
4. The *expenditure-minimizing bundle* to achive any level of utility for commodity i is given by:

$$h_i(p, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}$$

Important Identities

1. $e(\mathbf{p}, v(\mathbf{p}, m)) \equiv m$. The minimum expenditure necessary to reach utility $v(\mathbf{p}, m)$ is m.
2. $v(\mathbf{p}, e(\mathbf{p}, u)) \equiv u$. The maximum utiliity from income $e(\mathbf{p}, u)$ is u.
3. $x_i(\mathbf{p}, m) \equiv h_i(\mathbf{p}, v(\mathbf{p}, m))$. The Marshallian demand at income m is the same as the Hicksian demand at utility $v(\mathbf{p}, m)$.
4. $h_i(\mathbf{p}, u) \equiv x_i(\mathbf{p}, e(\mathbf{p}, u))$. The Hicksian demand at utility u is the same as the Marshallian demand at income $e(\mathbf{p}, u)$.

Roy's Identity

The Marshallian (ordinary) demand function is related to the indirect utility function as:

$$x_i(\mathbf{p}, m) = -\frac{\frac{\partial v(\mathbf{p},m)}{\partial p_i}}{\frac{\partial v(\mathbf{p},m)}{\partial m}}$$

provided that the functions are well-defined and $m > 0$.

Proof of Roy's Identity

Proof.

Suppose that x^* yields maximal utility of u^* at $(\mathbf{p}^*, m^*)$. Hence

$$\mathbf{x}(\mathbf{p}^*, m^*) \equiv \mathbf{h}(\mathbf{p}^*, u^*)$$

and

$$u^* = v(\mathbf{p}, e(\mathbf{p}, u^*)) \quad \forall \mathbf{p}$$

Differentiating this last relation:

$$0 = \frac{\partial v(\mathbf{p}^*, m^*)}{\partial p_i} + \frac{\partial v(\mathbf{p}^*, m^*)}{\partial m} \frac{\partial e(\mathbf{p}^*, u^*)}{\partial p_i}$$

we then have:

$$x_i(\mathbf{p}^*, m^*) \equiv h_i(\mathbf{p}^*, u^*) \equiv \frac{\partial e(\mathbf{p}^*, u^*)}{\partial p_i} = -\frac{\partial v(\mathbf{p}^*, m^*)/\partial p_i}{\partial v(\mathbf{p}^*, m^*)/\partial m}.$$

Money-Metric Utility

We define $m(\mathbf{p}, \mathbf{x})$ as follows:

$$m(\mathbf{p}, \mathbf{x}) \equiv e(\mathbf{p}, u(x))$$

This can be called a *money metric utility function*. $m(\mathbf{p}, \mathbf{x})$ is the minimum expenditure required when prices are p to achive the utility consistent with consumption bundle $\mathbf{x}$.
A dual form of this function is the *money metric indirect utility function*:

$$\mu(\mathbf{p}; \mathbf{q}, m) \equiv e(\mathbf{p}, v(\mathbf{q}, m)).$$

A Word Problem

Thomas lives in Ann Arbor where he currently spends 30% of his income on rent. He has an employment offer in Zürich which pays 50% more than he currently earns, but he is hesitant to take the job because rental rates in Zürich are three times higher than in Ann Arbor. Assuming that Thomas has CES preferences with elasticity of substitution σ*; on purely economic grounds, should he move?*

As is the case for all interesting questions in economics, the only good answer to this problem is "It depends.".

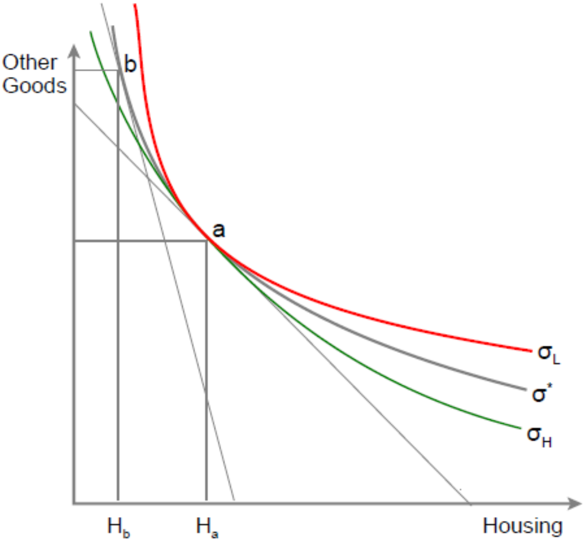
Intuition

Thomas's offer in Zürich does not pay him enough to live exactly the lifestyle that he enjoys in Ann Arbor, as he would need a 60% raise to cover rent and consumption. The elasticity of substitution is key. If it is high, he more willing substitutes consumption of goods and services for housing and thereby lowers his cost of living in Zürich. On the other hand, if the elasticity is low, he is "stuck in his ways", and the move is a bad idea.

Calibration to a Benchmark Equilibrium

We are given information about Thomas's choices in Ann Arbor. This information is essentially an observation of a *benchmark equilibrium*, consisting of the prevailing prices and quantities of goods demand. The benchmark equilibrium data together with assumptions about elasticities are used to evaluate Thomas's choices after a discrete change in the economic environment. The steps involved in solving this little textbook model are identical to those typically employed in applied general equilibrium analysis.

Graphical Representation



Preferences

The utility function:

$$U(C, H) = (\alpha C^{\rho} + (1-\alpha) H^{\rho})^{1/\rho}$$

Exponent ρ is defined by the elasticity of substitution, σ, as

$$\rho = 1 - 1/\sigma.$$

The model of consumer choice is:

$$\max U(C, H) \text{ s.t. } C + p_H H = M$$

Demand

Derivation of demand functions which solve the utility maximization problem involves solving two equations in two unknowns:

$$\frac{\partial U/\partial H}{\partial U/\partial C} = \frac{(1-\alpha)H^{\rho-1}}{\alpha C^{\rho-1}} = p_H;$$

hence

$$\frac{H}{C} = \left(\frac{1-\alpha}{\alpha \; p_H}\right)^{\sigma}$$

Substituting into the budget constraint, we have:

$$H = \frac{M}{p_H + \left(\frac{\alpha \; p_H}{1-\alpha}\right)^{\sigma}} = \frac{(1-\alpha)^{\sigma} M p_H^{-\sigma}}{\alpha^{\sigma} + (1-\alpha)^{\sigma} p_H^{1-\sigma}}$$

and

$$C = \frac{M}{1 + p_H \left(\frac{1-\alpha}{\alpha \; p_H}\right)^{\sigma}} = \frac{\alpha^{\sigma} M}{\alpha^{\sigma} + (1-\alpha)^{\sigma} p_H^{1-\sigma}}$$

Calibration

It is conventional in applied general equilibrium analysis to employ exogenous elasticities and calibrated value shares. If we follow this approach, σ is then exogenous and α is calibrated.

Choosing units so that the benchmark price of housing ($\bar{p}_H$) is unity, we have:

$$\theta = \bar{p}_H \bar{H} / \bar{M}$$

Substitute into the demand function:

$$1 + \left(\frac{\alpha}{1-\alpha} \right)^{\sigma} = \frac{\bar{M}}{\bar{H}} = \frac{1}{\theta};$$

and then solve for the preference parameter α:

$$\alpha = \frac{(1-\theta)^{1/\sigma}}{\theta^{1/\sigma} + (1-\theta)^{1/\sigma}}.$$

Money Metric Utility

Substitute for α in $U(C,H)$, and denoting the base year expenditure on other goods as $\bar{C} = (1-\theta)\bar{M}$, we have

$$U(C,H) = \kappa \left((1-\theta)^{1/\sigma} C^{\rho} + \theta^{1/\sigma} H^{\rho} \right)^{1/\rho}$$

where the κ is a constant which may take on any positive value without altering the preference ordering. It is convenient to assign this value to the benchmark expenditure, so that utility can be measured in money-metric units at benchmark prices.
Noting that $\theta^{1/\sigma} = \theta^{1-\rho}$, we then can write the utility function as:

$$\tilde{U}(C,H) = \bar{M} \left((1-\theta) \left(\frac{C}{\bar{C}} \right)^{\rho} + \theta \left(\frac{H}{\bar{H}} \right)^{\rho} \right)^{1/\rho}$$

Indirect Utility

Formally, we have:

$$V(p_H, M) = U(C(p_H, M), M(p_H, M)) = \frac{M}{\left(\alpha^\sigma + (1-\alpha)^\sigma p_H^{1-\sigma}\right)^{1/(1-\sigma)}}$$

In money-metric terms, we can use benchmark income to normalize the utility function:

$$\tilde{V}(p_H, M) = \frac{M}{(1-\theta+\theta p_H^{1-\sigma})^{1/(1-\sigma)}}$$

Demand Functions – Calibrated Share Form

$$H = \bar{H} \, \frac{\tilde{V}(p_H, M)}{\bar{M}} \left(\frac{p_U}{p_H}\right)^{\sigma} = \bar{H} \frac{M}{p_U \bar{M}} \left(\frac{p_U}{p_H}\right)^{\sigma}$$

$$C = \bar{C} \, \frac{\tilde{V}(p_H, M)}{\bar{M}} \left(\frac{p_U}{1}\right)^{\sigma} = \bar{C} \frac{M}{p_U \bar{M}} \left(\frac{p_U}{1}\right)^{\sigma}$$

where

$$p_U = \left(1 - \theta + \theta p_H^{1-\sigma}\right)^{1/(1-\sigma)}$$

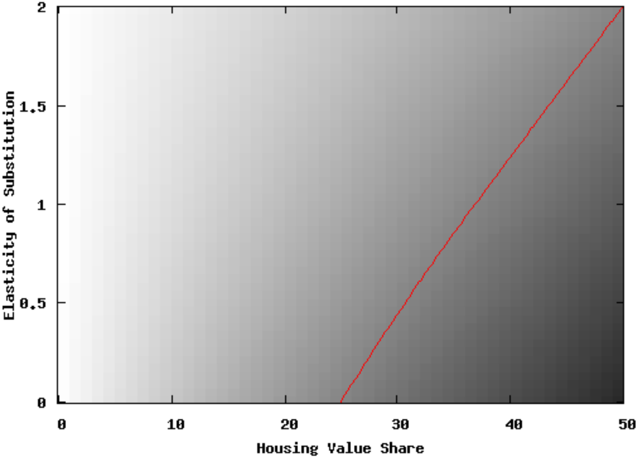
Should Thomas Move?

Thomas's welfare level in Zürich can be easily computed in money-metric terms as:

$$\tilde{V}(p_H = 3, M = 1.5) = \frac{1.5}{(0.7 + 0.3 \times 3^{1-\sigma})^{1/(1-\sigma)}}$$

This expression cannot (to my knowledge) be solved in closed form, however it is easily to solve using Excel. The critical value for σ is that which equates welfare in Zürich with welfare level in Ann Arbor, i.e. $\tilde{V} = 1$. The numerical value is found to be $\sigma^* = 0.441$. The general dependence of welfare on the θ and σ can be illustrated in a contour diagram.

Dependence of Welfare on Benchmark Shares and Elasticity



GNUPLOT Script

```
set auto
set style data lines
set xlabel "Housing Value Share"
set ylabel "Elasticity of Substitution"
set view map
set contour base
set xrange [0:50]
set yrange [0:2]
set cntrparam levels discrete 0
set pm3d
set palette gray positive
unset title
unset key
unset colorbox
unset clabel
set isosamples 51,50; set samples 51,50
set xtics
set ytics
unset surface
splot 1.50/(1-x/100+x/100*3**(1-y))**(1/(1-y))-1
```

CES Technology

A constant-elasticity-substitution production function can be defined as:

$$y = f(x) = \left(\sum_i a_i x_i^\rho \right)^{1/\rho}$$

where $a_i > 0 \quad \forall i$

The CES production function may alternatively be written as:

$$f(x) = \phi \left(\sum_i \alpha_i x_i^\rho \right)^{1/\rho}$$

where $\phi > 0$, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

Two Algebraic Facts

Two key algebraic identities are employed in this and subsequent derivations. For arbitrary real numbers, a, b and c, we have:

$$\left(a^b\right)^c = a^{bc}$$

and

$$a^b a^c = a^{b+c}$$

Equivalence of CES Functions

We need to show that there are values of ϕ and α_i in terms of a_i and ρ for which

$$\phi \left(\sum_i \alpha_i x_i^\rho \right)^{1/\rho} = \left(\sum_i a_i x_i^\rho \right)^{1/\rho}$$

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where $\phi > 0$, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.
For any $\beta > 0$

$$\begin{aligned} f(x) &= \left(\frac{\beta}{\beta} \sum_i a_i x_i^\rho \right)^{1/\rho} \\ &= \beta^{1/\rho} \left(\sum_i \frac{a_i}{\beta} x_i^\rho \right)^{1/\rho} \end{aligned}$$

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Let $\beta = \sum_i a_i$, $\alpha_i = a_i/\beta$ and $\phi = \beta^{1/\rho}$.

Returns to Scale

$f(x)$ exhibits constant returns to scale, i.e.

$$f(\lambda x) = \lambda f(x) \quad \forall \lambda > 0$$

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&= (\lambda^\rho)^{1/\rho} \left(\sum_i a_i x_i^\rho \right)^{1/\rho} \\
&= \lambda f(x)
\end{aligned}$$

Demand Fucntions

Cost-minimizing CES demand functions are:

$$x_i = \left(\frac{a_i c(p)}{p_i} \right)^{\sigma}$$

where

$$\sigma = \frac{1}{1 - \rho}$$

and

$$c(p) = \left(\sum_i a_i^{\sigma} p_i^{1-\sigma} \right)^{1-\sigma}$$

Deriving the Demand Function

The classical optimization problem is solved using the Lagrangian:

$$\mathcal{L} = \sum_i p_i x_i - \lambda \left(f(x) - 1 \right)$$

The Lagrange multiplier, λ, equals the marginal cost of output, c.[1] Hence, the first order condition for x_i reduces to:

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - c \frac{\partial f}{\partial x_i} = 0$$

[1] Note that because $f(x)$ exhibits constant returns to scale, this is also the *average* cost of production.

Deriving the Demand Function (cont.)

$$p_i = c \frac{\partial}{\partial x_i} \left(\sum_j a_j x_j^{\rho} \right)^{1/\rho}$$

Deriving the Demand Function (cont.)

$$\begin{aligned} p_i &= c \frac{\partial}{\partial x_i} \left(\sum_j a_j x_j^\rho \right)^{1/\rho} \\ &= c a_i x_i^{\rho - 1} \left(\sum_i a_i x_i^\rho \right)^{1/\rho - 1} \end{aligned}$$

Deriving the Demand Function (cont.)

$$
\begin{aligned}
p_i &= c\frac{\partial}{\partial x_i}\left(\sum_j a_j x_j^\rho\right)^{1/\rho} \\
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&= c a_i x_i^{\rho-1}\left(\sum_i a_i x_i^\rho\right)^{(1-\rho)/\rho}
\end{aligned}
$$

Deriving the Demand Function (cont.)

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&= c a_i x_i^{\rho-1}
\end{aligned}
$$

Deriving the Demand Function (cont.)

Letting $\sigma = 1/(1-\rho)$,

$$x_i^{-1/\sigma} = \frac{p_i}{ca_i}.$$

or

$$x_i = \left(\frac{a_i c}{p_i}\right)^{\sigma}.$$

Deriving the Demand Function (cont.)

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The cost function can be found by substituting $x_i(p, c)$ into the objective function:

$$c = \sum_i p_i x_i = \sum_i a_i^{\sigma} p_i^{1-\sigma} c^{\sigma} = c^{\sigma} \sum_i a_i^{\sigma} p_i^{1-\sigma}$$

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Hence:

$$c(p) = \left(\sum_i a_i^{\sigma} p_i^{1-\sigma}\right)^{1/(1-\sigma)}.$$

Shephard's Lemma

$$x_i(p) = \frac{\partial c(p)}{\partial p_i}$$

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Proof:

$$\frac{\partial c(p)}{\partial p_i} = \frac{\partial}{\partial p_i} \left(\sum_i a_i^{\sigma} p_i^{1-\sigma} \right)^{1/(1-\sigma)}$$

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Proof:

$$\begin{aligned}
\frac{\partial c(p)}{\partial p_i} &= \frac{\partial}{\partial p_i}\left(\sum_i a_i^\sigma p_i^{1-\sigma}\right)^{1/(1-\sigma)} \\
&= \left(\frac{a_i}{p_i}\right)^\sigma \left(\sum_i a_i^\sigma p_i^{1-\sigma}\right)^{1/(1-\sigma)-1}
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\end{aligned}$$

Allen-Uzawa Elasticity of Substitution

$$\sigma_{ij} \equiv \frac{\partial^2 c(p)}{\partial p_i \partial p_j} \frac{c(p)}{x_i x_j} = \sigma \quad \forall i \neq j$$

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Proof:

$$\begin{aligned} \frac{\partial^2 c(p)}{\partial p_i \partial p_j} &= \frac{\partial}{\partial p_j} \left(\frac{\partial c(p)}{\partial p_i} \right) \\ &= \frac{\partial x_i}{\partial p_j} = \frac{\partial}{\partial p_j} \left(\frac{a_i c(p)}{p_i} \right)^{\sigma} \end{aligned}$$

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&= \sigma \left(\frac{a_i}{p_i}\right)^{\sigma} c(p)^{\sigma-1} \frac{\partial c(p)}{\partial p_j} \quad \text{for } i \neq j
\end{aligned}$$

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&= \sigma \underbrace{\left(\frac{a_i c(p)}{p_i}\right)^{\sigma}}_{=x_i} \left(\frac{1}{c(p)}\right) \underbrace{\frac{\partial c(p)}{\partial p_j}}_{=x_j}
\end{aligned}$$

Allen-Uzawa Elasticity of Substitution

$$\sigma_{ij} \equiv \frac{\partial^2 c(p)}{\partial p_i \partial p_j} \frac{c(p)}{x_i x_j} = \sigma \quad \forall i \neq j$$

Proof:

$$\begin{aligned}
\frac{\partial^2 c(p)}{\partial p_i \partial p_j} &= \frac{\partial}{\partial p_j}\left(\frac{\partial c(p)}{\partial p_i}\right) \\
&= \frac{\partial x_i}{\partial p_j} = \frac{\partial}{\partial p_j}\left(\frac{a_i c(p)}{p_i}\right)^{\sigma} \\
&= \sigma \left(\frac{a_i}{p_i}\right)^{\sigma} c(p)^{\sigma-1} \frac{\partial c(p)}{\partial p_j} \quad \text{for } i \neq j \\
&= \sigma \underbrace{\left(\frac{a_i c(p)}{p_i}\right)^{\sigma}}_{=x_i} \left(\frac{1}{c(p)}\right) \underbrace{\frac{\partial c(p)}{\partial p_j}}_{=x_j} \\
&= \sigma \frac{x_i x_j}{c(p)}
\end{aligned}$$

Calibration

A firm produces output $\bar{y}$ with factor inputs $\bar{x}_i$ at factor prices $\bar{p}_i$. What values of a_i are consistent with this information, taking ρ (σ) as given?

Calibration (cont.)

CES coefficients a_i can be *calibrated* as:

$$a_i = \frac{\bar{p}_i(\bar{x}_i/\bar{y})^{1-\rho}}{\bar{c}}$$

where $\bar{c}$ is benchmark unit cost:

$$\bar{c} = \frac{\sum_i \bar{p}_i \bar{x}_i}{\bar{y}}$$

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Proof:
The demand function derived above is that which minimizes the cost of producing one unit of output. With constant returns to scale, the cost minimizing factor demands associated with output level y are proportional to the unit demand, i.e.

$$x_i(p, y) = x_i(p)y = \frac{\partial c(p)}{\partial p_i} y$$

Inverting the Demand Function

Given $\bar{c}$, we can invert the factor demand function to determine a_i:

$$\bar{x}_i = \left(\frac{a_i c(\bar{p})}{\bar{p}_i} \right)^{\sigma} \bar{y},$$

hence

$$a_i = \frac{\bar{p}_i (\bar{x}_i / \bar{y})^{1/\sigma}}{\bar{c}}$$

which is our result given $\rho = 1 - 1/\sigma$.

Unit Functions

A *unit function* is a function which evaluates to unity at a reference point. If the reference point is $\bar{x} \in R^n$, and $f(x)$ is a unit function, then $f(x)|_{x=\bar{x}} = 1$.

The Calibrated Share Form

The calibrated form of a CES unit cost function can be written as:

$$c = \bar{c} \left(\sum_i \theta_i \left(\frac{p_i}{\bar{p}_i} \right)^{1-\sigma} \right)^{1/(1-\sigma)}$$

where $\bar{c}$ is the benchmark unit cost and θ_i is the benchmark value share of the ith input:

$$\theta_i = \frac{\bar{p}_i \bar{x}_i}{\sum_j \bar{p}_j \bar{x}_j}$$

Deriving The Calibrated Share Form

Proof:

$$c(p) = \left(\sum_i a_i^\sigma p_i^{1-\sigma} \right)^{1/(1-\sigma)}$$

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$$\begin{aligned} c(p) &= \left(\sum_i a_i^\sigma p_i^{1-\sigma} \right)^{1/(1-\sigma)} \\ &= \left(\sum_i \left(\frac{\bar{p}_i (\bar{x}_i/\bar{y})^{1/\sigma}}{\bar{c}} \right)^\sigma p_i^{1-\sigma} \right)^{1/(1-\sigma)} \end{aligned}$$

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c(p) &= \left(\sum_i a_i^\sigma p_i^{1-\sigma} \right)^{1/(1-\sigma)} \\
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&= \left(\sum_i \frac{\bar{p}_i^\sigma \bar{x}_i}{\bar{c}^\sigma \bar{y}} p_i^{1-\sigma} \right)^{1/(1-\sigma)}
\end{aligned}
$$

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&= \left(\sum_i \frac{\bar{p}_i^{\sigma} \bar{x}_i}{\bar{c}^{\sigma} \bar{y}} p_i^{1-\sigma} \right)^{1/(1-\sigma)} \\
&= \left(\bar{c}^{1-\sigma} \sum_i \frac{\bar{p}_i \bar{x}_i}{\bar{c} \bar{y}} \left(\frac{p_i}{\bar{p}_i} \right)^{1-\sigma} \right)^{1/(1-\sigma)}
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&= \bar{c} \left(\sum_i \theta_i \left(\frac{p_i}{\bar{p}_i} \right)^{1-\sigma} \right)^{1/(1-\sigma)}
\end{aligned}
$$

Calibrated Demand Function

The compensated demand function can be written as

$$x_i = \bar{x}_i \left(\frac{c(p)\bar{p}_i}{\bar{c} \; p_i} \right)^{\sigma} \frac{y}{\bar{y}}$$

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Proof:

$$x_i(p, y) \quad = \quad \left(\frac{a_i c(p)}{p_i} \right)^{\sigma} y$$

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Proof:

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Calibrated Production Function

$$y = \bar{y} \left(\sum_i \theta_i \left(\frac{x_i}{\bar{x}_i} \right)^{\rho} \right)^{1/\rho}$$

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&= \left(\sum_i \frac{\bar{p}_i \bar{x}_i}{\bar{c} \bar{y}} \; \frac{\bar{x}_i^{1/\sigma - 1}}{\bar{y}^{1/\sigma - 1}} \; x_i^\rho \right)^{1/\rho} \\
&= \left(\sum_i \theta_i \; \frac{\bar{x}_i^{-\rho}}{\bar{y}^{-\rho}} \; x_i^\rho \right)^{1/\rho}
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&= \left(\sum_i \theta_i \; \frac{\bar{x}_i^{-\rho}}{\bar{y}^{-\rho}} \; x_i^{\rho}\right)^{1/\rho} \\
&= \bar{y}\left(\sum_i \theta_i \left(\frac{x_i}{\bar{x}_i}\right)^{\rho}\right)^{1/\rho}
\end{aligned}$$

Essential Inputs: $\sigma < 1$

A CES technology with $\sigma < 1$ is calibrated to a reference point with $\bar{x}_i$, $\bar{y}$ and $\bar{p}_i$. When $\sigma < 1$, the minimum demand for good *i per unit of output* is given by:

$$\underline{x}_i = \theta_i^{\sigma/(1-\sigma)} \frac{\bar{x}_i}{\bar{y}}$$

where θ_i is the benchmark value share of the *i*th input, as defined above.

Essential Inputs: $\sigma < 1$

Consider points on the *unit* isoquant, i.e.

$$\left(\sum_j a_j x_j^{\rho}\right)^{1/\rho} = 1$$

or

$$\sum_j a_j x_j^{\rho} = 1$$

Essential Inputs: $\sigma < 1$

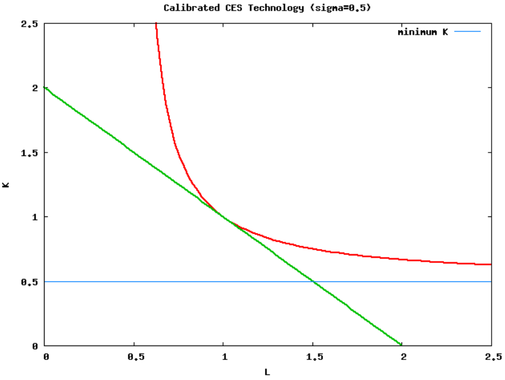
The minimum input of factor i is realized when all other inputs expand without bound. Take this limit and require that the input of good i is sufficient to remain on the unit isoquant, hence:

$$\lim_{\substack{x_j \to \infty \\ \forall j \neq i}} \left(\sum_j a_j x_j^\rho \right) = a_i x_i^\rho = 1$$

Substitute the calibrated value of a_i to obtain:

$$\underline{x}_i = \theta_i^{\sigma/(1-\sigma)} \frac{\bar{x}_i}{\bar{y}}$$

Essential Inputs: $\sigma < 1$




```
reset
sigma = 0.5
theta = 0.5
rho(sigma) = 1 - 1/sigma
f(x,sigma) = ((1-theta*x**(1-1/sigma))/(1-theta))**(1/(1-1/sigma))
set xrange[0:2.5]
set yrange[0:2.5]
set xlabel 'L'
set ylabel 'K'
set title 'Calibrated CES Technology (sigma=0.5)'
plot f(x,0.5) lw 2 lc 1 notitle , \
     f(x,4.0) lw 2 lc 3 notitle , \
```

Inessential Inputs: $\sigma > 1$

We define input i as *essential* if

$$\lim_{x_i \to 0} f(x) = 0$$

When $\sigma > 1$, then none of the inputs are essential.

Inessential Inputs: Proof

Examine the unit isoquant:

$$f(x) = \left(\sum_j a_j x_j^\rho \right)^{1/\rho} = 1$$

or

$$\sum_j a_j x_j^\rho = 1$$

If $\sigma > 1$, then $\rho > 0$.
It follows immediately that only one input need be provided at a positive level. If $x_j = 0 \; \forall j \neq \hat{i}$, then

$$\sum_j a_j x_j^\rho = a_{\hat{i}} x_{\hat{i}}^\rho$$

and we can choose the single input to maintain feasiblity:

$$x_{\hat{i}} = a_{\hat{i}}^{-1/\rho}.$$

Corner Solutions: NOT

When factor prices are finite and nonzero, it is never cost effective to let any input fall to zero.

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When factor prices are finite and nonzero, it is never cost effective to let any input fall to zero.

Proof:
First, note that when $\sigma < 1$, all inputs are essential. We therefore only be concerned with cases in which $\sigma > 1$. In this case, however, the isoquant is *tangent to but does not intersect* the axis.

Corner Solutions: NOT

When factor prices are finite and nonzero, it is never cost effective to let any input fall to zero.

Proof:
First, note that when $\sigma < 1$, all inputs are essential. We therefore only be concerned with cases in which $\sigma > 1$. In this case, however, the isoquant is *tangent to but does not intersect* the axis.
When prices are non-zero and finite, the unit demand for x_i is

$$x_i = \left(\frac{a_i c(p)}{p_i} \right)^{\sigma}$$

when is never zero when $c(p) > 0$ and $p_i < \infty$.