A Logit technology for general equilibrium analysis

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Abstract: We propose a new Logit-based cost function as a foundation for the technology assumed in economic analysis. The technology departs from other Logit applications in that it does not require an outside good. This makes it internally consistent in general-equilibrium applications. In its non-separable nested form the proposed technology is shown to be flexible in calibration to any consistent local observation of the Slutsky matrix, and it remains convex globally on the open price simplex. The technology is "regular flexible." We demonstrate the technology in the context of the context of radical reduction in fossil energy use. The implied costs under the GE-Logit technology are compared with the widely adopted constant-elasticity-of-substitution (CES) technology calibrated to an identical local benchmark.

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1. Introduction

Standard constant-elasticity-of-substitution (CES) based technologies are regular, but impose severe restrictions on *observable* local cross-price responses. Perroni and Rutherford (1995) propose extending the CES structure to include nonseparabilities across nests. The resulting Nonseparable Nested CES (NNCES) functional form is flexible to any consistent local observation of price responses and maintains regularity globally. Further, Perroni and Rutherford (1998) show that other proposed flexible functional forms can be problematic for global inference because they quickly lose regularity away from the local point of observation. Having the NNCES form is useful, but the lack of alternatives might restrict the range of empirical inference.

In this paper we propose a new unit cost function that might be used as a building block for an alternative regular-flexible technology. The cost function is based on the multinomial logit model. This formulation departs from other logit applications

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in that the only payments are to inputs at market prices, with the associated random input costs internalized as an aggregate productivity adjustments. Thus, no outside-good (money) payments are needed to cover the random-cost component critical to input choice. This makes the technology internally consistent in general-equilibrium application. In its non-separable nested form the proposed technology is shown to be flexible in calibration to any consistent local observation of the Slutsky matrix, and it remains convex globally on the open price simplex. The technology is *regular flexible* as defined by Perroni and Rutherford (1995), and is therefore a viable candidate as an alternative to the NNCES technology.

We offer a demonstration of our proposed technology in the context of the Net-Zero challenge, which postulates a radical reduction in fossil energy use. We formulate a transparent general equilibrium with output produced by two primary inputs, capital and labor, and a third intermediate input, fossil energy. The model is calibrated to observed input shares. In an exercise similar to Hogan and Mann (1977) we illustrate that beliefs about the local elasticities are critical in assessing the costs associated with significant fossil-energy reductions. We add a systematic comparison of the new Logit based technology with the CES standard. We conclude that regularity restricts the range of outcomes on a sizable "inner domain," but globally the GE Logit model departs from the CES model.

2. Theory

The Canonical Logit Model

A production process involves numerous tasks. Any task can be completed with any two or more factors of production. When the manager assigns factor i to a task she realizes unit cost:

$$c_i = \alpha_i + \mu\tilde{\epsilon}$$

where α_i is the composite cost associated with factor i (the same for all tasks), μ is a positive constant, and $\tilde{\epsilon}$ is a gumbel distributed random variable with mean γ and variace $\frac{\pi^2}{6}$ variance. Variations in $\tilde{\epsilon}$ reflect heterogeneity of the productivity of factors. Each factor has the same random variation in cost for all tasks. Despite random variations in factor cost there is no explicit uncertainty in factor choice. That is, the cost of doing any task k with factor i is known *before* the factor assignments are made. Given

$$c_{ik} = \alpha_i + \mu\epsilon_{ik},$$

the least cost production plan assigns factors of production to tasks, and it can be formulated as a trivial linear programming problem:

$$\min \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{m} c_{ik}x_{ik}\right) \tag{1}$$

subject to

$$\sum_i x_{ik} = 1 \quad \forall k = \{1, \ldots, n\}$$

$$x_{ik} \geq 0$$

where m denotes the number of factors of production and n is the number of tasks involved in the production process (in this discrete approximation[1]).

The least cost factor is assigned to each task. The ex-ante probability that factor i is chosen is

$$\pi_i = \text{Prob}\left[c_i = \min_{j=1,\ldots,m} c_j\right] \quad i = 1, \ldots, m.$$

Assuming that the ϵ's are identically, independently Gumbel distributed, the choice probability becomes the (multinomial) logit. The fraction of tasks assigned to factor i is:

$$\pi_i = \frac{e^{-\alpha_i/\mu}}{\sum_j e^{-\alpha_j/\mu}} \quad i = 1, \ldots, m$$

The mean cost of production is[2]

$$V = -\mu\left[\log\left(\sum_i e^{-\alpha_i/\mu}\right) + \gamma\right]$$

where γ is the Euler-Mascheroni constant (≈ 0.5772).

Calibration: Partial Equilibrium

We now reinterpret the factor cost coefficient α_i to account for factor prices (p_i) and a reference equilibrium in which the fraction of all tasks assigned to factor i is θ_i at reference factor prices $\bar{p}_i = 1$. Consider the following assignment:

$$\alpha_i = p_i + \mu\left(\gamma - \log(\theta_i)\right).$$

When α_i is defined this way, the unit cost of completing an individual task using factor i is:

$$c_i = p_i - \mu(\log(\theta_i) + \tilde{\epsilon} - \gamma),$$

[1] In the continuous model we let $n \to \infty$.
[2] Need to include the derivation for this. I can't find my notes – integration by parts or something. The vaue of γ essentially corresponds to the term which can't be obtained in closed form.

and the share of tasks assigned to factor i is

$$\pi_i = \frac{\theta_i e^{(1-p_i)/\mu}}{\sum_j \theta_j e^{(1-p_j)/\mu}} \quad i = 1, \ldots, m,$$

and the production cost is:

$$V = 1 - \mu \log\left(\sum_i \theta_i e^{(1-p_i)/\mu}\right).$$

At reference prices $\bar{p}_i = 1 \quad \forall i$, $\pi_i(p) = \theta_i$, and $V = 1$ (for any value of μ).

Calibration: General Equilibrium

Input costs in the partial equilibrium model are denominated in units of factors (at prices p_i) and the nominal values of the associated random costs. These play the role of an "outside good" in the model formulation, portraying inputs coming from markets which are not included explicitly in the model. If we are going to formulate a version of the logit model which is useful for general equilibrium modeling, we need to have a closed model. In simple terms, we need a model in which the only inputs to the production process are factors i. How then do we denominate random variations in factor cost? The iceberg model from international trade provides one solution.

Working with a discrete approximation, the linear programming representation analogous to the partial equilibrium model (1) is as follows:

$$\min \frac{1}{n} \sum_{k=1}^{n} \left(\sum_{i=1}^{m} \alpha_i x_{ik} \right)$$

subject to

$$y = 1 + \frac{\mu}{n} \sum_{ik} \epsilon_{ik} x_{ik} \quad \perp p_y \tag{2}$$

$$\sum_i x_{ik} = y \quad \forall k = \{1, \ldots, n\} \quad \perp \mu_k$$

$$x_{ik} \geq 0$$

In the discrete iceberg model random variations in factor cost appear on the right-hand side of constraint (2). Random variations in cost of factor are denominated in units of output from the very same sector, and for this reason *gross output* (y) differs from *net output* (1). Substitution choices on the input side affect aggregate productivity. [3] In the continuous model, the unit cost of sectoral output (a composite of many

[3] This linear program can be solved iteratively. See notes in `simulate.gms`. When n is large,

requisite tasks) may be denoted as v. Using v to denominate random variations in production cost, the unit cost of completing a task using factor i is

$$c_i = p_i - v\ \mu\ (\log(\theta_i) + \tilde{\epsilon} - \gamma)$$

The shares of tasks assigned to factor i are then:

$$\pi_i = \frac{\theta_i e^{(1-p_i/v)/\mu}}{\sum_j \theta_j e^{(1-p_j/v)/\mu}},$$

and the aggregate cost of production is:

$$V(p,v) = v\left[1 - \mu \log\left(\sum_i \theta_i e^{(1-p_i/v)/\mu}\right)\right],$$

which equals unity when $p_i = v = 1\ \ \forall i$.

Unlike the constant-elasticity-of-substitution (CES) cost function which can be calculated *explicitly* from factor prices, the iceberg logit cost function is *implicitly* determined by the zero profit condition:

$$V(p,v) = v,$$

an equation which simplifies to:

$$\sum_i \theta_i e^{(1-p_i/v)/\mu} = 1 \tag{3}$$

Provided that the unit cost function satisfies the zero profit condition, the choice probability for factor i can be written as:

$$\pi_i = \theta_i e^{(1-p_i/v)/\mu}$$

v must be computed simultaneously with factor demand which are determined in part by the share of tasks which are assigned to each factor as well as by the number of tasks which must be completed to produce a unit of net output. Overall productivity depends on the extent to which primary factors are substituted against non-factor inputs. A change in the ratio of v to factor costs results in higher or lower productivity, as part of sectoral output may be diverted as an intermediate production input. Intuitively, we can compute factor demands as:

$$x_i = \left(\frac{v}{\sum_j p_j \pi_j}\right)\pi_i = \theta_i \frac{e^{(1-p_i/v)/\mu}}{\phi}$$

in which

$$\phi \equiv \sum_j (p_j/v)\pi_j = \sum_j \theta_j \frac{p_j}{v} e^{(1-p_j/v)/\mu}$$

the LP can be slow (see `calibIogit.gms`), but the iterative solution takes only a couple of iterations. We need to explain how this works.

Comparative Statics

In the logit model, the cost function is defined implicitly by (3) which can be written:

$$f(p,v) = \sum_i \theta_i e^{(1-p_i/v)/\mu} = 1$$

Linearizing around a solution to this equation, considering perturbations of p_i and v which maintain feasibility, we set the total derivative of $f()$ to zero:

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial p_i}\mathrm{d}p_i + \frac{\partial f}{\partial v}\mathrm{d}v = 0. \tag{4}$$

The requisite partial derivatives are:

$$\frac{\partial f}{\partial p_i} = -\frac{\theta_i/\mu}{v} e^{(1-p_i/v)/\mu} = -\frac{\pi_i}{\mu v},$$

and

$$\frac{\partial f}{\partial v} = \sum_i \frac{\theta_i p_i/\mu}{v^2} e^{(1-p_i/v)/\mu} = \frac{\phi}{\mu v},$$

Substituting into (4) provides Shephard's lemma: the derivative of unit cost with respect to factor price i equals the compensated demand for that factor:

$$\frac{\mathrm{d}v}{\mathrm{d}p_i} = \frac{-\partial f/\partial p_i}{\partial f/\partial v} = \frac{\theta_i e^{(1-p_i/v)/\mu}}{\sum_j \theta_j (p_j/v) e^{(1-p_j/v)/\mu}} = \frac{\pi_i}{\phi} = x_i$$

The matrix of second order gradients of unit cost with respect to factor prices is the Slutsky matrix. We can compute this directly by differentiation of x_i with respect to p_j. The calculation involves a few partial derivatives:

$$\frac{\partial \pi_i}{\partial p_j} = \begin{cases} -\frac{\pi_i}{\mu v} & i = j \\ 0 & i \neq j \end{cases},$$

$$\frac{\partial \pi_i}{\partial v} = \frac{p_i}{\mu v^2}\pi_i,$$

$$\frac{\partial \phi}{\partial p_i} = \frac{\pi_i}{v} + \frac{p_i}{v}\frac{\partial \pi_i}{\partial p_i} = \frac{\pi_i}{v}\left(1 - \frac{p_i}{\mu v}\right),$$

and (defining $\hat{\phi} = \sum_i (p_i/v)^2 \pi_i$):

$$\frac{\partial \phi}{\partial v} = \frac{1}{v}\left(\hat{\phi}/\mu - \phi\right).$$

Slutsky terms may then be computed directly using the chain rule:

$$
\begin{aligned}
\left.\frac{\mathrm{d}x_i}{\mathrm{d}p_j}\right|_{i\neq j} &= \frac{\mathrm{d}}{\mathrm{d}p_j}\frac{\pi_i}{\phi} \\
&= \frac{1}{\phi}\frac{\mathrm{d}\pi_i}{\mathrm{d}p_j} - \frac{\pi_i}{\phi^2}\frac{\mathrm{d}\phi}{\mathrm{d}p_j} \\
&= \frac{1}{\phi}\left(\frac{\partial\pi_i}{\partial p_j} + \frac{\partial\pi_i}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}p_i}\right) - \frac{\pi_i}{\phi^2}\left(\frac{\partial\phi}{\partial p_j} + \frac{\partial\phi}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}p_j}\right) \\
&= \frac{1}{\phi}\left(0 + \frac{p_i}{\mu v^2}\pi_i x_j\right) - \frac{\pi_i}{\phi^2}\left(\frac{\pi_j}{v}\left(1 - \frac{p_j}{\mu v}\right) + \frac{1}{v}\left(\frac{\hat{\phi}}{\mu} - \phi\right)x_j\right)
\end{aligned}
$$

Hence, off-diagonal terms in the Slutsky matrix are given by:

$$
\left.\frac{\mathrm{d}x_i}{\mathrm{d}p_j}\right|_{i\neq j} = \frac{1}{\mu}\;\frac{x_i x_j}{v}\;\left[\frac{p_i + p_j}{v} - \frac{\hat{\phi}}{\phi}\right]
$$

and the diagonal term is:

$$
\frac{\mathrm{d}x_i}{\mathrm{d}p_i} = \frac{x_i}{\mu v}\left[x_i\left(\frac{2p_i}{v} - \frac{\hat{\phi}}{\phi}\right) - 1\right]
$$

The compensated own and cross-price elasticities of demand are then

$$
\epsilon_{ij} = \frac{\partial x_i}{\partial p_j}\frac{p_i}{x_j} = \begin{cases} \frac{1}{\mu}\left(\tilde{\theta}_i\left(\frac{2p_i}{v} - \frac{\hat{\phi}}{\phi}\right) - \frac{p_i}{v}\right) & i = j \\[2ex] \frac{\tilde{\theta}_i}{\mu}\left(\frac{p_i + p_j}{v} - \frac{\hat{\phi}}{\phi}\right) & i \neq j \end{cases}
$$

where $\tilde{\theta}_i = p_i x_i / v$ is the value share of good i at prices p.

Comparison with CES

We have produced a calibration of the logit demand function which matches up with a constant elasticity of substitution model. The CES model is formulated as a technology-constrained cost minimization problem:[4]

$$
\min \sum_j p_j x_j
$$

subject to

$$
\left(\sum_j \theta_j \left(\frac{x_j}{\bar{x}_j}\right)^{\rho}\right)^{1/\rho} = 1
$$

where $\rho = 1 - 1/\sigma$ is defined by the Allen-Uzawa elasticity of substitution, σ. The cost minimizing demand for factor i is:

[4] Here I use the calibrated share form of the CES demand system, simplified to the case in which reference prices are unity. Equations for the non-unitary case for the most part involve a global replacement of p_i by $p_i / \bar{p}_i$.

$$x_j = \bar{x}_j \left(\frac{v}{p_j}\right)^{\sigma}$$

where

$$v = \left(\sum_j \theta_j p_j^{1-\sigma}\right)^{1/(1-\sigma)}$$

and θ_j remains the benchmark value share of good j.

Off-diagonal terms in the Slutsky matrix are given by:

$$\left.\frac{\mathrm{d}x_i}{\mathrm{d}p_j}\right|_{i\neq j} = \sigma \frac{x_i x_j}{v}$$

The Allen-Uzawa elasticity of substitution, σ, is a free parameter which describes the compensated own and cross-price elasticities of demand:

$$\epsilon_{ij} = \frac{\partial x_i}{\partial p_j}\frac{p_i}{x_j} = \begin{cases} \sigma(\tilde{\theta}_i - 1) & i = j \\ \sigma\tilde{\theta}_i & i \neq j \end{cases}$$

where $\tilde{\theta}_i$ is the value share of good i at prices p.

Equivalence of the logit and CES demand systems at the benchmark point implies

$$\mu = \frac{1}{\sigma}$$

Boundary Solutions

One challenge with the CES function is that the boundaries of the price simplex lie outside the domain of the cost function when $\sigma < 0$. When $\sigma > 0$, the cost function can be evaluated with one or more prices equal to zero, but demands for those goods whose prices fall to zero are undefined. In the case of the iceberg logit model, the demand function for commodity i can be evaluated when $p_i = 0$, but the unit cost function may be undefined.

Suppose that $p_i = 0 \quad \forall i \in \mathscr{I}_0$. The unit cost function v then satisfied the following equation:

$$\sum_{j\notin\mathscr{I}_0} \theta_j e^{(1-p_j/v)\sigma} = 1 - e^{\sigma} \sum_{i\in\mathscr{I}_0} \theta_i$$

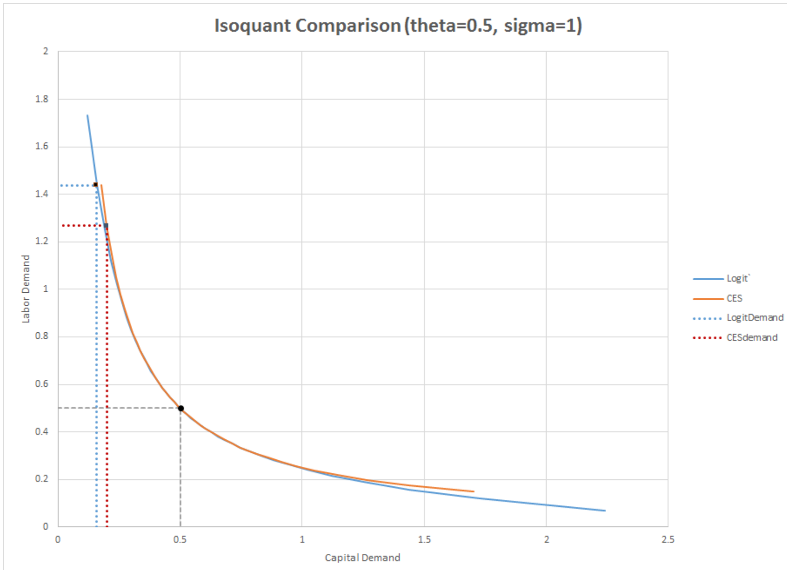
This equation has no real solution v when

$$\sigma > \sigma^{\max} = \frac{1}{\log\left(\sum_{i\in\mathscr{I}_0} \theta_i\right)}$$

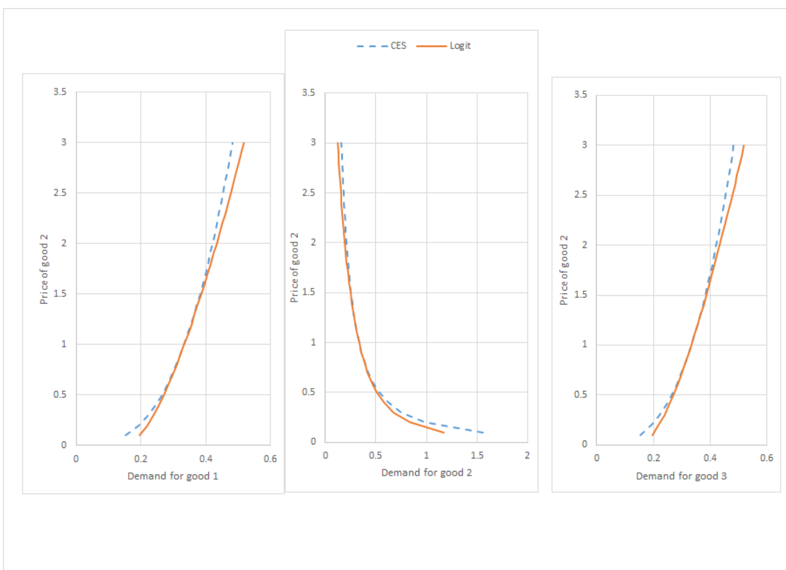
Table 1. Measured Inner Domain for a Symmetric Benchmark

	Symmetric Value Shares: $\theta = (0.33, 0.33, 0.33)$															
	Compensated				Allen-Uzawa				Morishima				Shadow			
$\sigma^A =$	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4
CES	73	100	18	2	100	100	100	100	100	100	56	29	100	100	67	29
LOGIT	13	18	13	2	24	24	13	13	51	35	13	2	51	35	13	2
	Assymetric Value Shares: $\theta = (0.35, 0.6, 0.05)$															
	Compensated				Allen-Uzawa				Morishima				Shadow			
$\sigma^A =$	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4	0.5	1	2	4
CES	84	100	49	13	100	100	100	100	95	100	75	27	100	100	95	58
LOGIT	25	27	16	7	20	20	20	11	33	35	18	7	27	33	18	11

Figure 1. Isoquants – $\sigma = 1$

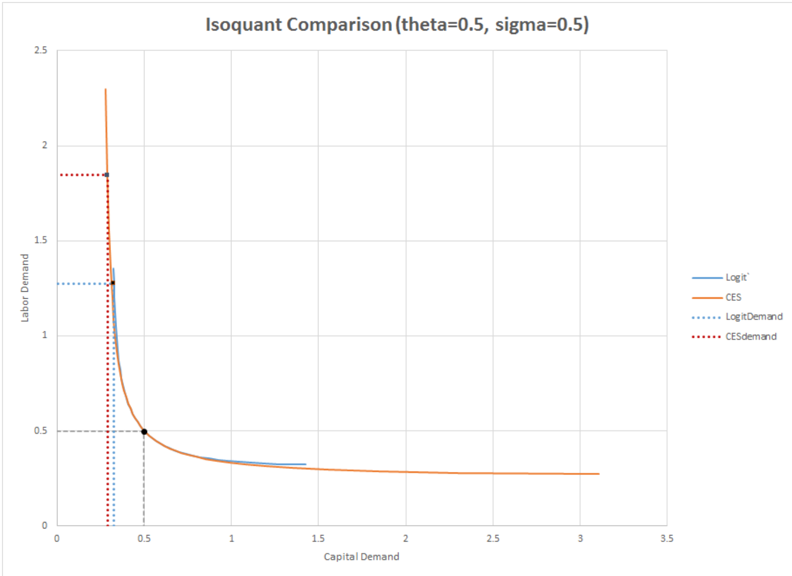


(a) Isoquant

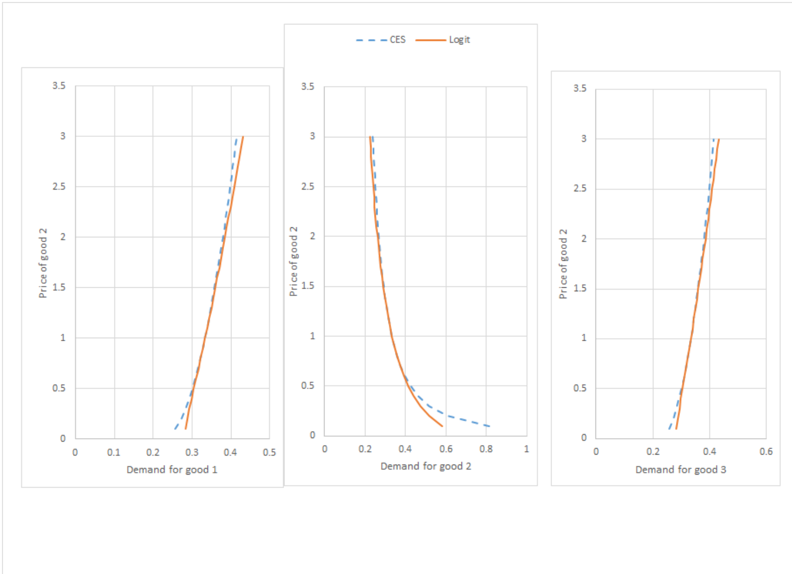


(b) Demand Functions

Figure 2. Isoquants – $\sigma = 1/2$



(a) Isoquant



(b) Demand Functions

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